

TRANSFERS OF METABELIAN p -GROUPS

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ABSTRACT. Explicit expressions for the transfers V_i from a metabelian p -group G of coclass $\text{cc}(G) = 1$ to its maximal normal subgroups M_i ($1 \leq i \leq p+1$) are derived by means of relations for generators. The expressions for the exceptional case $p = 2$ differ significantly from the standard case of odd primes $p \geq 3$. In both cases the transfer kernels $\text{Ker}(V_i)$ are calculated and the principalisation type of the metabelian p -group is determined, if G is realised as the Galois group $\text{Gal}(\mathbb{F}_p^2(K)|K)$ of the second Hilbert p -class field $\mathbb{F}_p^2(K)$ of an algebraic number field K . For certain metabelian 3-groups G with abelianisation G/G' of type $(3, 3)$ and of coclass $\text{cc}(G) = r \geq 3$, it is shown that the principalisation type determines the position of G on the coclass graph $\mathcal{G}(3, r)$ in the sense of Eick and Leedham-Green.

1. INTRODUCTION

In [19] we have used the theory of dihedral fields of degree $2p$ with a prime $p \geq 3$ to show for a quadratic base field $K = \mathbb{Q}(\sqrt{D})$ with p -class group $\text{Cl}_p(K) = \text{Syl}_p(\text{Cl}(K))$ of type (p, p) and second p -class group $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ of maximal class that the entire p -class group of K becomes principal in at least p unramified cyclic extension fields $N_i|K$ of relative degree p .

In the present paper we generalise this result for an arbitrary base field K with p -class group $\text{Cl}_p(K)$ of type (p, p) and second p -class group G of maximal class. The proof consists of computing the images and kernels of the transfers V_i from G to its maximal normal subgroups M_i with $1 \leq i \leq p+1$ and thus determining the principalisation type of K . Our statements are mainly expressed in group theoretical form, using transfer types instead of principalisation types, since the realisation of the p -group G as Galois group $\text{Gal}(\mathbb{F}_p^2(K)|K)$ of the second Hilbert p -class field of an algebraic number field K , though being fundamental for the number theoretical applications, is inessential for proving our results.

Section 2 is devoted to the metabelian p -groups G of maximal class for an arbitrary prime $p \geq 2$. A p -group G of order $|G| = p^m$ and nilpotency class $\text{cl}(G) = m - 1$ is called of maximal class or of coclass $\text{cc}(G) = m - \text{cl}(G) = 1$. That is a CF-group with first factor $G/\gamma_2(G)$ of type (p, p) and cyclic further factors $\gamma_j(G)/\gamma_{j+1}(G)$ of order p for $2 \leq j \leq m - 1$, where we denote by $\gamma_j(G)$ the members of the lower central series of G . Relations for the generators x, y of a metabelian p -group $G = \langle x, y \rangle$ of maximal class have been given by N. Blackburn [6] and more generally by R. J. Miech [21] and are recalled at the beginning of section 2.

In subsection 2.1 we derive explicit expressions for the transfers $V_i : G/\gamma_2(G) \rightarrow M_i/\gamma_2(M_i)$ from G to the maximal self-conjugate subgroups M_1, \dots, M_{p+1} with the aid of commutator calculus and the presentations given by Miech. The expressions obtained for the exceptional case $p = 2$ differ significantly from the uniform standard case of odd primes $p \geq 3$.

Subsection 2.2 introduces the group theoretical concepts of singulets and multiplets of transfer types, which describe the transfer kernels. In the intermediate subsection 2.3 we use the reciprocity law of E. Artin [1, 2] to show how the group theoretical statements can be applied to algebraic number fields K with p -class group $\text{Cl}_p(K)$ of type (p, p) . The *second p -class group* of such a field K , that is the Galois group $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ of the second Hilbert p -class field $\mathbb{F}_p^2(K)$ of K ,

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is a metabelian p -group with abelianisation $G/\gamma_2(G)$ of type (p, p) [19]. In the exceptional case $p = 2$ it is always of maximal class.

By the computation of the transfer kernels $\text{Ker}(V_i)$ of G we determine the principalisation type of the p -class group $\text{Cl}_p(K)$ of K in the $p + 1$ unramified cyclic extensions $N_i|K$ of relative degree p in subsection 2.4. For $p \geq 5$, our results are very similar to those of B. Nebelung [23, p.202, Satz 6.9] for $p = 3$. However, in the exceptional case $p = 2$ our precise principalisation types permit more explicit statements than H. Kisilevsky [16, p.273, Th.2] or E. Benjamin and C. Snyder [4, p.163, §2], since the cohomology $H^0(\text{Gal}(N_i|K), \text{Cl}_p(N_i))$ of the p -class groups $\text{Cl}_p(N_i)$ with respect to the relative Galois groups $\text{Gal}(N_i|K)$ [15] only yields the coarse distinction of the conditions (A) and (B) in the sense of O. Taussky [26, p.435].

Transfer images and kernels of metabelian 3-groups G with abelianisation $G/\gamma_2(G)$ of type $(3, 3)$ and of coclass $\text{cc}(G) \geq 2$ are investigated in section 3, based on presentations given by Nebelung [23].

In subsection 3.2 we show for certain 3-groups G of coclass $\text{cc}(G) = r \geq 3$ that it can be decided by means of the parity of the index of nilpotency of the second 3-class group G of a quadratic number field $K = \mathbb{Q}(\sqrt{D})$, if G is represented by a terminal or an internal node on the finitely many directed and rooted trees of isomorphism classes of metabelian 3-groups of coclass r with abelianisation of type $(3, 3)$ in the sense of Nebelung [23, p.181 ff], which are infinite subgraphs of the coclass graph $\mathcal{G}(3, r)$ in the sense of Ascione, Leedham-Green, et al. [3, 17, 9, 8].

At the end of sections 2 and 3 we present a complete overview of all transfer types of metabelian p -groups with abelianisation of type (p, p) for $p = 2$ and $p = 3$.

2. TRANSFERS OF A METABELIAN p -GROUP OF MAXIMAL CLASS

With an arbitrary prime $p \geq 2$, let G be a metabelian p -group of order $|G| = p^m$ and nilpotency class $\text{cl}(G) = m - 1$, where $m \geq 3$. Then G is of maximal class and the commutator factor group $G/\gamma_2(G)$ of G is of type (p, p) [6, 21]. The lower central series of G is defined recursively by $\gamma_1(G) = G$ and $\gamma_j(G) = [\gamma_{j-1}(G), G]$ for $j \geq 2$.

The centraliser $\chi_2(G) = \{g \in G \mid [g, u] \in \gamma_4(G) \text{ for all } u \in \gamma_2(G)\}$ of the two-step factor group $\gamma_2(G)/\gamma_4(G)$, that is,

$$\chi_2(G)/\gamma_4(G) = \text{Centraliser}_{G/\gamma_4(G)}(\gamma_2(G)/\gamma_4(G)),$$

is the biggest subgroup of G such that $[\chi_2(G), \gamma_2(G)] \leq \gamma_4(G)$. It is characteristic, contains the commutator group $\gamma_2(G)$, and coincides with G , if and only if $m = 3$. Let the isomorphism invariant $k = k(G)$ of G be defined by

$$[\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G),$$

where $k = 0$ for $m = 3$, $0 \leq k \leq m - 4$ for $m \geq 4$, and $0 \leq k \leq \min\{m - 4, p - 2\}$ for $m \geq p + 1$, according to Miech [21, p.331].

Suppose that generators of $G = \langle x, y \rangle$ are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, and $y \in \chi_2(G) \setminus \gamma_2(G)$. We define the main commutator $s_2 = [y, x] \in \gamma_2(G)$ and the higher commutators $s_j = [s_{j-1}, x] = s_{j-1}^{x-1} \in \gamma_j(G)$ for $j \geq 3$. In the sequel, we only need two relations for p th powers of the generators x and y of G , when we calculate explicit images of the transfers of G ,

$$(1) \quad x^p = s_{m-1}^w \quad \text{and} \quad y^p \prod_{\ell=2}^p s_{\ell}^{\binom{p}{\ell}} = s_{m-1}^z \quad \text{with exponents} \quad 0 \leq w, z \leq p - 1,$$

according to Miech [21, p.332, Th.2, (3)]. Blackburn uses the notation $\delta = w$ and $\gamma = z$ for these relational exponents [6, p.84, (36), (37)].

Additionally, the group G satisfies relations for p th powers of the higher commutators,

$$s_{j+1}^p \prod_{\ell=2}^p s_{j+\ell}^{\binom{p}{\ell}} = 1 \quad \text{for } 1 \leq j \leq m - 2,$$

and the main commutator relation of Miech [21, p.332, Th.2, (2)],

$$(2) \quad [y, s_2] = \prod_{r=1}^k s_{m-r}^{a(m-r)} \in [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G),$$

with exponents $0 \leq a(m-r) \leq p-1$ for $1 \leq r \leq k$, $a(m-k) > 0$. Blackburn restricts his investigations to $k \leq 2$ and uses the notation $\beta = a(m-1)$ and $\alpha = a(m-2)$ [6, p.82, (33)].

By $G_a^{(m)}(z, w)$ we denote the representative of an isomorphism class of metabelian p -groups G of maximal class and of order $|G| = p^m$, which satisfies the relations (2) and (1) with a fixed system of exponents $a = (a(m-k), \dots, a(m-1))$ and w and z .

The maximal normal subgroups M_i of G contain the commutator group $\gamma_2(G)$ of G as a normal subgroup of index p and thus are of the shape $M_i = \langle g_i, \gamma_2(G) \rangle$. We define an order by $g_1 = y$ and $g_i = xy^{i-2}$ for $2 \leq i \leq p+1$. The commutator groups $\gamma_2(M_i)$ are of the general form $\gamma_2(M_i) = \langle s_2, \dots, s_{m-1} \rangle^{g_i^{-1}}$, according to [19, Cor.3.1.1], and in particular

$$(3) \quad \begin{aligned} \gamma_2(M_1) &= \begin{cases} 1, & \text{if } k = 0, \\ \gamma_{m-k}(G), & \text{if } k \geq 1, \end{cases} \\ \gamma_2(M_i) &= \gamma_3(G) \quad \text{for } 2 \leq i \leq p+1. \end{aligned}$$

For an arbitrary fixed index $1 \leq i \leq p+1$ we select an element $h \in G \setminus M_i$ and denote the p th trace element (*Spur*) of h in the group ring $\mathbb{Z}[G]$ by $S_p(h) = \sum_{\ell=1}^p h^{\ell-1}$.

Taking into consideration that $G^p < \gamma_2(G) < M_i$, the *transfer* (*Verlagerung*) $V_i = V_{G, M_i}$ from G to M_i is given by

$$(4) \quad V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto \begin{cases} g^p \gamma_2(M_i), & \text{if } g \in G \setminus M_i, \\ g^{S_p(h)} \gamma_2(M_i), & \text{if } g \in M_i, \end{cases}$$

according to E. Artin [2, p.50], H. Hasse [12, p.171, VII], or K. Miyake [22, p.296 ff].

2.1. Images of the transfers. Since the elementary abelian bicyclic p -group G of type (p, p) represents a degenerate case of the metabelian p -groups of maximal class, which cannot be excluded generally in the subsequent number theoretical applications, we begin by determining its transfers.

Theorem 2.1. *With an arbitrary prime $p \geq 2$, let G be the elementary abelian bicyclic p -group of type (p, p) and order $|G| = p^m$, $m = 2$ with its $p+1$ cyclic subgroups as maximal normal subgroups M_1, \dots, M_{p+1} . Let*

$$V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto V_i(g\gamma_2(G))$$

be the transfer from G to M_i for $1 \leq i \leq p+1$.

Then the images of all transfers are trivial,

$$V_i(g\gamma_2(G)) = 1 \quad \text{for } g \in G \text{ and } 1 \leq i \leq p+1.$$

Proof. Like any elementary abelian p -group, G is of exponent p . Further, all commutator groups $\gamma_2(M_i) = 1$ are trivial. In formula (4), with an arbitrary element $h \in G \setminus M_i$, we consequently obtain the triviality of the explicit image of each transfer,

$$V_i(g\gamma_2(G)) = \begin{cases} g^p \gamma_2(M_i) = 1, & \text{if } g \in G \setminus M_i, \\ g^{S_p(h)} \gamma_2(M_i) = g^{\sum_{\ell=1}^p h^{\ell-1}} = \prod_{\ell=1}^p g^{h^{\ell-1}} = \prod_{\ell=1}^p g = g^p = 1, & \text{if } g \in M_i, \end{cases}$$

for $1 \leq i \leq p+1$. □

After this degenerate case we treat the non-abelian p -groups G of maximal class with abelian commutator group $\gamma_2(G)$ and commutator factor group $G/\gamma_2(G)$ of type (p, p) for the uniform standard case of odd primes $p \geq 3$.

Theorem 2.2. *With an odd prime $p \geq 3$, let $G = \langle x, y \rangle$ be a metabelian p -group of maximal class of order $|G| = p^m$, where $m \geq 3$. Suppose that the generators of G are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, $y \in \chi_2(G) \setminus \gamma_2(G)$, and the relations (1) with exponents $0 \leq w, z \leq p-1$ are satisfied. Let the maximal normal subgroups M_1, \dots, M_{p+1} be ordered by $M_1 = \langle y, \gamma_2(G) \rangle$ and $M_i = \langle xy^{i-2}, \gamma_2(G) \rangle$ for $2 \leq i \leq p+1$. Finally, let*

$$V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto V_i(g\gamma_2(G))$$

denote the transfer from G to M_i for $1 \leq i \leq p+1$.

Assume that the cosets $g\gamma_2(G) \in G/\gamma_2(G)$ are represented in the shape $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $0 \leq j, \ell \leq p-1$, then the images of the transfers are generally given by

$$V_i(x^j y^\ell \gamma_2(G)) = s_{m-1}^{wj+z\ell} \gamma_2(M_i) \quad \text{for } 1 \leq i \leq p+1.$$

With the explicit form (3) of the commutator groups $\gamma_2(M_i)$, they are given by

$$\begin{aligned} V_1(x^j y^\ell \gamma_2(G)) &= \begin{cases} s_{m-1}^{wj+z\ell} \cdot 1, & \text{if } [\chi_2(G), \gamma_2(G)] = 1, \quad k = 0, \quad m \geq 3, \quad p \geq 3, \\ 1 \cdot \gamma_{m-1}(G), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G), \quad k = 1, \quad m \geq 5, \quad p \geq 3, \\ 1 \cdot \gamma_{m-k}(G), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G), \quad k \geq 2, \quad m \geq 6, \quad p \geq 5, \end{cases} \\ V_i(x^j y^\ell \gamma_2(G)) &= \begin{cases} s_2^{wj+z\ell} \cdot 1, & \text{if } m = 3, \\ 1 \cdot \gamma_3(G), & \text{if } m \geq 4, \end{cases} \quad \text{for } 2 \leq i \leq p+1. \end{aligned}$$

Proof. By means of formula (4), we calculate the explicit image $V_i(g\gamma_2(G))$ of the transfer, if the element g is represented by the generators x, y of G in the shape $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $0 \leq j, \ell \leq p-1$.

- (1) We start with the first maximal normal subgroup $M_1 = \langle y, \gamma_2(G) \rangle$, for which $x \in G \setminus M_1$ and $y \in M_1$ and thus

$$\begin{aligned} V_1(x\gamma_2(G)) &= x^p \gamma_2(M_1) = s_{m-1}^w \gamma_2(M_1), \\ V_1(y\gamma_2(G)) &= y^{S_p(x)} \gamma_2(M_1) = y^{\sum_{j=1}^p x^{j-1}} \gamma_2(M_1). \end{aligned}$$

We distinguish two cases, according to the invariant k of G .

If $k = 0$, then $\gamma_2(M_1) = 1$, M_1 is an abelian normal subgroup, and we can use symbolic powers with the difference $X = x - 1 \in \mathbb{Z}[G]$ in the exponents:

$$V_1(y\gamma_2(G)) = y^{\sum_{j=1}^p \binom{p}{j} X^{j-1}} = \prod_{j=1}^p (y^{X^{j-1}})^{\binom{p}{j}} = y^p \prod_{j=2}^p s_j^{\binom{p}{j}} = s_{m-1}^z.$$

However, if $k \geq 1$, then $\gamma_2(M_1) = \gamma_{m-k}(G) \geq \gamma_{m-1}(G) > 1$. Here we must derive a special case of a commutator formula by Miech [21, p.338, Lem.2],

$$[y, x^j] = [y, x] S_j(x) = s_2^{\sum_{r=1}^j x^{r-1}} = s_2^{\sum_{r=1}^j \binom{j}{r} X^{r-1}} = \prod_{r=1}^j (s_2^{X^{r-1}})^{\binom{j}{r}} = \prod_{r=1}^j s_{r+1}^{\binom{j}{r}},$$

and we obtain $V_1(y\gamma_2(G)) =$

$$= y^{\sum_{j=0}^{p-1} x^j} \gamma_2(M_1) = \prod_{j=0}^{p-1} y^{x^j} \gamma_2(M_1) = \prod_{j=0}^{p-1} (y[y, x^j]) \gamma_2(M_1) = \prod_{j=0}^{p-1} (y \prod_{r=1}^j s_{r+1}^{\binom{j}{r}}) \gamma_2(M_1).$$

Now we observe that y commutes with all elements s_ℓ modulo $\gamma_2(M_1)$,

$$s_\ell y = y s_\ell [s_\ell, y] \equiv y s_\ell \pmod{\gamma_2(M_1)},$$

since $[s_\ell, y] \in \langle s_2, \dots, s_{m-1} \rangle^{y-1} = \gamma_2(G)^{y-1} = \gamma_2(M_1)$ for all $\ell \geq 2$, and we obtain

$$V_1(y\gamma_2(G)) = (\prod_{j=0}^{p-1} y) (\prod_{j=0}^{p-1} \prod_{r=1}^j s_{r+1}^{\binom{j}{r}}) \gamma_2(M_1).$$

An interchange of the order of the factors in the double product yields

$$\begin{aligned} V_1(y\gamma_2(G)) &= y^p \prod_{r=1}^{p-1} \prod_{j=r}^{p-1} s_{r+1}^{\binom{j}{r}} \gamma_2(M_1) \\ &= y^p \prod_{t=2}^p \prod_{j=t-1}^{p-1} s_t^{\binom{j}{t-1}} \gamma_2(M_1) = y^p \prod_{t=2}^p s_t^{\sum_{j=t-1}^{p-1} \binom{j}{t-1}} \gamma_2(M_1). \end{aligned}$$

Using the index transformation $s = t-1$, $\ell = j-s$, $n = p-1-s$ and the following well-known identity for binomial coefficients,

$$\sum_{\ell=0}^n \binom{s+\ell}{s} = \binom{s+n+1}{s+1} \quad \text{with } s, n \geq 0,$$

we finally obtain

$$V_1(y\gamma_2(G)) = y^p \prod_{t=2}^p s_t^{\binom{p}{t}} \gamma_2(M_1) = s_{m-1}^z \gamma_2(M_1).$$

- (2) The second maximal normal subgroup is $M_2 = \langle x, \gamma_2(G) \rangle$, whence $x \in M_2$ and $y \in G \setminus M_2$. Therefore,

$$V_2(x\gamma_2(G)) = x^{S_p(y)}\gamma_2(M_2) = x^{\sum_{j=0}^{p-1} y^j} \gamma_3(G) = \prod_{j=0}^{p-1} x^{y^j} \gamma_3(G) = \prod_{j=0}^{p-1} (x[x, y^j])\gamma_3(G).$$

Now we use a congruence modulo $\gamma_3(G)$ for commutators of powers,

$$[x, y^j] \equiv [x, y]^j \pmod{\gamma_3(G)},$$

which is due to Blackburn [6, p.49, Th.1.4], and we observe that

$$[x, y]x = x[x, y] \cdot [[x, y], x] \equiv x[x, y] \pmod{\gamma_3(G)}.$$

$V_2(x\gamma_2(G)) = \prod_{j=0}^{p-1} (x[x, y]^j)\gamma_3(G) = (\prod_{j=0}^{p-1} x)(\prod_{j=0}^{p-1} [x, y]^j)\gamma_3(G) = x^p(s_2^{-1})^{\sum_{j=0}^{p-1} j}\gamma_3(G) = x^p s_2^{-\binom{p}{2}}\gamma_3(G) = x^p\gamma_3(G) = s_{m-1}^w\gamma_3(G)$, since $(\gamma_2(G) : \gamma_3(G)) = p$ and thus $s_2^p \in \gamma_3(G)$. Here it is essential that p divides the binomial coefficient $\binom{p}{2}$, which is only correct for odd $p \geq 3$. Further, we have

$V_2(y\gamma_2(G)) = y^p\gamma_2(M_2) = y^p \prod_{j=2}^p s_j^{\binom{p}{j}} \gamma_3(G) = s_{m-1}^z\gamma_3(G)$, since $s_j \in \gamma_3(G)$ for $j \geq 3$ and $s_2^{\binom{p}{2}} \in \gamma_3(G)$, provided that $p \geq 3$.

- (3) All the other maximal normal subgroups M_i with $3 \leq i \leq p+1$ can be treated in a uniform way. Since $M_i = \langle xy^{i-2}, \gamma_2(G) \rangle$, we have $x \in G \setminus M_i$ and $y \in G \setminus M_i$ and thus

$$V_i(x\gamma_2(G)) = x^p\gamma_2(M_i) = s_{m-1}^w\gamma_3(G), \text{ similarly as for } i = 1,$$

$$V_i(y\gamma_2(G)) = y^p\gamma_2(M_i) = s_{m-1}^z\gamma_3(G), \text{ similarly as for } i = 2.$$

According to [6, p.82], the possible maximum of the invariant k depends on m and p . \square

Now we separately analyse the exceptional case $p = 2$ with its irregular expressions for the transfers.

Theorem 2.3. *Let $G = \langle x, y \rangle$ be a metabelian 2-group of maximal class and of order $|G| = 2^m$, where $m \geq 3$. Suppose that the generators of G are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, $y \in \chi_2(G) \setminus \gamma_2(G)$, and the relations (1) are satisfied with exponents $0 \leq w, z \leq 1$. Let the maximal normal subgroups M_1, \dots, M_3 be ordered by $M_1 = \langle y, \gamma_2(G) \rangle$, $M_2 = \langle x, \gamma_2(G) \rangle$, and $M_3 = \langle xy, \gamma_2(G) \rangle$. Assume that*

$$V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto V_i(g\gamma_2(G))$$

denotes the transfer from G to M_i for $1 \leq i \leq 3$.

If the cosets $g\gamma_2(G) \in G/\gamma_2(G)$ are represented by $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $0 \leq j, \ell \leq 1$, then the images of the transfers V_i reveal an irregular behavior for $2 \leq i \leq 3$ and are given by

$$\begin{aligned} V_1(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} \cdot 1, \\ V_2(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} s_2^{-(j+\ell)} \gamma_3(G) = \begin{cases} s_2^{(w-1)j+(z-1)\ell} \cdot 1, & \text{if } m = 3, \\ s_2^{-(j+\ell)} \gamma_3(G), & \text{if } m \geq 4, \end{cases} \\ V_3(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} s_2^{-\ell} \gamma_3(G) = \begin{cases} s_2^{wj+(z-1)\ell} \cdot 1, & \text{if } m = 3, \\ s_2^{-\ell} \gamma_3(G), & \text{if } m \geq 4. \end{cases} \end{aligned}$$

Proof. According to [5, p.23, Th.1.2] and [10, Ch.5, Th.4.5], G is isomorphic to either a dihedral group of order 2^m , $m \geq 3$, or a generalised quaternion group of order 2^m , $m \geq 3$, or a semi-dihedral group of order 2^m , $m \geq 4$. In each case, we have $s_j = y^{(-2)^{j-1}}$ for $j \geq 3$. Thus $\gamma_2(G) = \langle s_2, \dots, s_{m-1} \rangle = \langle s_2 \rangle = \langle y^2 \rangle$, and the representation of the three maximal normal subgroups M_i of G becomes simply

$$\begin{aligned} M_1 &= \langle y, \gamma_2(G) \rangle = \langle y, s_2 \rangle = \langle y \rangle \text{ (cyclic),} \\ M_2 &= \langle x, \gamma_2(G) \rangle = \langle x, s_2 \rangle, \\ M_3 &= \langle xy, \gamma_2(G) \rangle = \langle xy, s_2 \rangle. \end{aligned}$$

The commutator groups $\gamma_2(M_i)$ are of the form

$$\begin{aligned}\gamma_2(M_1) &= 1, \text{ since } k = 0, \\ \gamma_2(M_i) &= \gamma_3(G) \text{ for } 2 \leq i \leq 3.\end{aligned}$$

Similarly as in the proof of theorem 2.2, we select an element $h \in G \setminus M_i$ and denote by $S_2(h) = \sum_{j=1}^2 h^{j-1} = 1 + h \in \mathbb{Z}[G]$ the second trace element of h , for an arbitrary fixed index $1 \leq i \leq 3$. Then the transfer from G to M_i is given by

$$V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto \begin{cases} g^2\gamma_2(M_i), & \text{if } g \in G \setminus M_i, \\ g^{1+h}\gamma_2(M_i), & \text{if } g \in M_i, \end{cases}$$

according to formula (4). By means of these formulas we calculate the explicit image $V_i(g\gamma_2(G))$ of the transfer, where the element g is represented by the generators x, y of G in the shape $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $0 \leq j, \ell \leq 1$.

We partially obtain irregular images of the transfers, since the main commutator s_2 appears only in its first power in the Blackburn relation $y^2 s_2 = s_{m-1}^z$ for the second power of y , and s_2 is not contained in $\gamma_3(G)$. Irregularities only occur for V_2 and V_3 , but not for the abelian maximal normal subgroup M_1 . The images of the generators are

$$\begin{aligned}V_1(y) &= y^{1+x}\gamma_2(M_1) = yx^{-1}yx \cdot 1 = y^2y^{-1}x^{-1}yx \cdot 1 = y^2s_2 \cdot 1 = s_{m-1}^z \cdot 1, \\ V_1(x) &= x^2\gamma_2(M_1) = s_{m-1}^w \cdot 1, \\ V_2(y) &= y^2\gamma_2(M_2) = s_{m-1}^z s_2^{-1} \gamma_3(G) \text{ (irregular)}, \\ V_2(x) &= x^{1+y}\gamma_2(M_2) = xy^{-1}xy\gamma_3(G) = x^2x^{-1}y^{-1}xy\gamma_3(G) = x^2s_2^{-1}\gamma_3(G) = s_{m-1}^w s_2^{-1} \gamma_3(G) \text{ (irregular)}, \\ V_3(y) &= y^2\gamma_2(M_3) = s_{m-1}^z s_2^{-1} \gamma_3(G) \text{ (irregular)}, \\ V_3(x) &= x^2\gamma_2(M_3) = s_{m-1}^w \gamma_3(G).\end{aligned}$$

□

2.2. Singulets and multiplets of transfer types. For the kernel of the transfer

$$V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i), \quad g\gamma_2(G) \mapsto V_i(g\gamma_2(G))$$

from a p -group G with abelian commutator group $\gamma_2(G)$ and commutator factor group $G/\gamma_2(G)$ of type (p, p) to one of its maximal normal subgroups M_i , where $1 \leq i \leq p+1$, we have $p+3$ possibilities,

either $\text{Ker}(V_i) = 1$ or $\text{Ker}(V_i) = M_j/\gamma_2(G)$ with $1 \leq j \leq p+1$ or $\text{Ker}(V_i) = G/\gamma_2(G)$.

However, in subsection 2.4 it will turn out that a transfer V_i is never injective, that is, the transfer kernel $\text{Ker}(V_i) > 1$ is always non-trivial, provided that G is of maximal class.

Consequently, there remain $p+2$ possible *singulets of transfer types* $\varkappa(i)$ for each individual maximal normal subgroup $M_i < G$:

- either a *total* transfer with two-dimensional kernel,

$$\text{Ker}(V_i) = G/\gamma_2(G), \text{ denoted by the symbol } \varkappa(i) = 0,$$

- or a *partial* transfer with one-dimensional kernel,

$$\text{Ker}(V_i) = M_j/\gamma_2(G) \text{ for some } 1 \leq j \leq p+1, \text{ denoted by the symbol } \varkappa(i) = j.$$

According to Taussky [26, p.435], we also have a coarser distinction:

- condition (A),

$$\text{Ker}(V_i) \cap M_i/\gamma_2(G) > 1,$$

since either $\text{Ker}(V_i) = G/\gamma_2(G)$ with $\varkappa(i) = 0$ for a total transfer or $\text{Ker}(V_i) = M_i/\gamma_2(G)$ with $\varkappa(i) = i$ for a partial transfer with *fixed point* of \varkappa ,

- condition (B),

$$\text{Ker}(V_i) \cap M_i / \gamma_2(G) = 1,$$

since $\text{Ker}(V_i) = M_j / \gamma_2(G)$ with $\varkappa(i) = j \neq i$ for a partial transfer without fixed point, where the exact value of j remains unknown.

The singulets are combined to a *multiplet of transfer types* for the family of all $p+1$ maximal normal subgroups $M_i < G$ with $1 \leq i \leq p+1$,

$$\varkappa = (\varkappa(1), \dots, \varkappa(p+1)) \in [0, p+1]^{p+1}.$$

The number of total transfers is an invariant $\nu = \nu(G)$ of the p -group G :

Definition 2.1. Let $0 \leq \nu \leq p+1$ be the number $\nu = \#\{1 \leq i \leq p+1 \mid \text{Ker}(V_i) = G / \gamma_2(G)\}$ of maximal normal subgroups M_i of G , for which the transfer V_i from G to M_i is total.

We call a multiplet $\varkappa = (\varkappa(1), \dots, \varkappa(p+1))$ of transfer types *partial*, if $\varkappa(i) \neq 0$ for all $1 \leq i \leq p+1$, that is, if $\nu = 0$, and otherwise *total*.

The orbit $\varkappa^{S_{p+1}}$ of a multiplet $\varkappa \in [0, p+1]^{p+1}$ under the operation $\varkappa^\pi = \pi_0^{-1} \circ \varkappa \circ \pi$ of the symmetric group S_{p+1} of degree $p+1$ is an invariant of the p -group G , which is independent from the order of the normal subgroups M_i , and is called the *transfer type* $\varkappa(G)$ of G . Here we assume that the extension π_0 of $\pi \in S_{p+1}$ to $[0, p+1]$ fixes the zero.

2.3. Extension and principalisation of ideal classes. At the present position it is adequate to turn to the number theoretical applications of our purely group theoretical results.

For an arbitrary prime $p \geq 2$, the second p -class group [19], that is the Galois group $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ of the second Hilbert p -class field $\mathbb{F}_p^2(K)$, of an algebraic number field K with p -class group $\text{Cl}_p(K)$ of type (p, p) is a metabelian p -group G with abelianisation $G / \gamma_2(G)$ of type (p, p) . The reason for this fact is that, according to the reciprocity law of Artin [1] and Galois correspondence, the commutator group

$$\gamma_2(G) = \text{Gal}(\mathbb{F}_p^2(K)|\mathbb{F}_p^1(K)) \simeq \text{Cl}_p(\mathbb{F}_p^1(K))$$

is abelian as the p -class group of the first Hilbert p -class field $\mathbb{F}_p^1(K)$ of K and the commutator factor group

$$G / \gamma_2(G) = \text{Gal}(\mathbb{F}_p^2(K)|K) / \text{Gal}(\mathbb{F}_p^2(K)|\mathbb{F}_p^1(K)) \simeq \text{Gal}(\mathbb{F}_p^1(K)|K) \simeq \text{Cl}_p(K)$$

is of type (p, p) .

By the Galois correspondence $M_i = \text{Gal}(\mathbb{F}_p^2(K)|N_i)$, the maximal normal subgroups M_1, \dots, M_{p+1} of G are associated with the $p+1$ unramified cyclic extensions N_1, \dots, N_{p+1} of K of relative degree p , which are represented by the norm class groups $\text{Norm}_{N_i|K}(\text{Cl}_p(N_i))$ as subgroups of index p in the p -class group $\text{Cl}_p(K)$ of K , according to [1]. The abelianisations of the M_i ,

$$M_i / \gamma_2(M_i) = \text{Gal}(\mathbb{F}_p^2(K)|N_i) / \text{Gal}(\mathbb{F}_p^2(K)|\mathbb{F}_p^1(N_i)) \simeq \text{Gal}(\mathbb{F}_p^1(N_i)|N_i) \simeq \text{Cl}_p(N_i),$$

are isomorphic to the p -class groups of the N_i , by [1].

In algebraic number theory, we are interested in the type of the principalisation of ideal classes of $\text{Cl}_p(K)$ in the p -class groups $\text{Cl}_p(N_i)$ [19, 5.3]. Therefore, we investigate the kernel of the *class extension* homomorphisms

$$\mathfrak{j}_{N_i|K} : \text{Cl}_p(K) \longrightarrow \text{Cl}_p(N_i), \quad \mathfrak{a}\mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_{N_i})\mathcal{P}_{N_i},$$

where \mathcal{P} denotes the group of principal ideals and \mathcal{O} the maximal order. According to Hilbert's theorem 94 [13, p.279], the class extension $\mathfrak{j}_{N_i|K}$ has a non-trivial kernel $\text{Ker}(\mathfrak{j}_{N_i|K}) > 1$. We define the *multiplet* $\varkappa \in [0, p+1]^{p+1}$ of *principalisation types* of K for $1 \leq i \leq p+1$ by

$$\text{Ker}(\mathfrak{j}_{N_i|K}) = \text{Norm}_{N_{\varkappa(i)}|K}(\text{Cl}_p(N_{\varkappa(i)})),$$

if $1 \leq \varkappa(i) \leq p+1$, that is, for *partial* principalisation, and we put $\varkappa(i) = 0$ for *total* principalisation, $\text{Ker}(\mathfrak{j}_{N_i|K}) = \text{Cl}_p(K)$. The *principalisation type* $\varkappa(K)$ of K is defined as the orbit $\varkappa^{S_{p+1}}$ of the multiplet \varkappa under the operation $\varkappa^\pi = \pi_0^{-1} \circ \varkappa \circ \pi$ of the symmetric group S_{p+1} of degree $p+1$, where we assume that the extension π_0 of $\pi \in S_{p+1}$ to $[0, p+1]$ fixes the zero.

The extent of total principalisation is expressed by an invariant $\nu = \nu(K)$ of the field K :

Definition 2.2. Let $0 \leq \nu \leq p+1$ be the number $\nu = \#\{1 \leq i \leq p+1 \mid \text{Ker}(j_{N_i|K}) = \text{Cl}_p(K)\}$ of unramified cyclic extension fields N_i of K of relative degree p , in which the entire p -class group $\text{Cl}_p(K)$ of K becomes principal (cfr. [7, capitulation number, p.1230]).

According to Artin [2] (see also Miyake [22, p.297, Cor.]), the following commutative diagram establishes the connection between the number theoretical extensions $j_{N_i|K}$ of p -class groups and the group theoretical transfers $V_i = V_{G, M_i}$ from the abelianisation of the second p -class group $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ of K to the abelianisations of its maximal normal subgroups $M_i = \text{Gal}(\mathbb{F}_p^2(K)|N_i)$ with $1 \leq i \leq p+1$.

$$\begin{array}{ccccc}
 & & j_{N_i|K} & & \\
 & \text{Cl}_p(K) & \longrightarrow & \text{Cl}_p(N_i) & \\
 \text{Artin isomorphism} & \downarrow & & \downarrow & \text{Artin isomorphism} \\
 & G/\gamma_2(G) & \longrightarrow & M_i/\gamma_2(M_i) & \\
 & V_{G, M_i} & & &
 \end{array}$$

Due to the commutativity of this diagram, the following number theoretical and group theoretical concepts correspond to each other: principalisation kernels and transfer kernels, $\text{Ker}(j_{N_i|K}) \simeq \text{Ker}(V_i)$ for $1 \leq i \leq p+1$, norm class groups and cyclic subgroups of $G/\gamma_2(G)$, $\text{Norm}_{N_i|K}(\text{Cl}_p(N_i)) \simeq M_i/\gamma_2(M_i)$ for $1 \leq i \leq p+1$, multiplets of principalisation types, multiplets of transfer types, and their orbits, $\varkappa(K) = \varkappa(G)$, and finally the invariants $\nu(K) = \nu(G)$ describing the total principalisation and the total transfer.

Remark 2.1. For an arbitrary prime $p \geq 2$ and an arbitrary base field K with p -class group of type (p, p) and second p -class group $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ of maximal class, Hilbert's theorem 94 [13], that is the non-injectivity of the class extension homomorphisms $j_{N_i|K}$, is a consequence of the non-injectivity of the transfers V_i , which will be proved in the theorems 2.4, 2.5, and 2.6. For second p -class groups G of non-maximal class with $G/\gamma_2(G)$ of type (p, p) , which can occur only for $p \geq 3$, the non-injectivity of the transfers V_i has been proved by Nebelung for $p = 3$ [23, p.208, Satz 6.14].

2.4. Kernels of the transfers. By means of the expressions for the transfers, which have been determined in subsection 2.1, we are now able to calculate the transfer kernels $\text{Ker}(V_i)$ for $1 \leq i \leq p+1$ and the transfer type $\varkappa(G)$ of the metabelian p -group G of maximal class. We begin with the degenerate case of the elementary abelian p -group of type (p, p) .

Theorem 2.4. *Let $p \geq 2$ be an arbitrary prime and G the elementary abelian bicyclic p -group of type (p, p) and of order $|G| = p^m$, $m = 2$ with its $p+1$ cyclic subgroups as maximal normal subgroups M_1, \dots, M_{p+1} . Then the following statements hold.*

- (1) *The kernels of the transfers $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ for $1 \leq i \leq p+1$ are all two-dimensional of type (p, p) ,*

$$\text{Ker}(V_i) = G/\gamma_2(G), \quad \varkappa(i) = 0.$$

- (2) *The multiplet $\varkappa = (\varkappa(1), \dots, \varkappa(p+1))$ of the transfer types of G is given by*

$$\varkappa = (\overbrace{0, \dots, 0}^{p+1 \text{ times}}), \quad \nu = p+1.$$

Proof. By theorem 2.1, all images of the transfers are trivial, $V_i(g\gamma_2(G)) = 1$, for all elements $g \in G$ and all $1 \leq i \leq p+1$. Therefore we have $\text{Ker}(V_i) = G/\gamma_2(G)$ for $1 \leq i \leq p+1$. \square

Now we come to the uniform standard case of a metabelian p -group of maximal class with an odd prime $p \geq 3$.

Theorem 2.5. *Let $p \geq 3$ be an odd prime and G a metabelian p -group of maximal class of order $|G| = p^n$ and class $\text{cl}(G) = m - 1$ where $n = m \geq 3$. Suppose that generators x, y of G are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, $y \in \chi_2(G) \setminus \gamma_2(G)$, and the relations (1) with exponents $0 \leq w, z \leq p - 1$ are satisfied. In the case of $m = 3$ and the extra special p -group $G \simeq G_0^{(3)}(0, 1)$ of exponent p^2 , let y be of order p . Let the maximal normal subgroups M_1, \dots, M_{p+1} of G be ordered by $M_1 = \langle y, \gamma_2(G) \rangle$ and $M_i = \langle xy^{i-2}, \gamma_2(G) \rangle$ for $2 \leq i \leq p + 1$. Then, in particular $M_1 = \chi_2(G)$, if $m \geq 4$, and the following statements hold.*

(1) *The kernels of the transfers $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ for $1 \leq i \leq p + 1$ are given by*

$$\begin{aligned} \text{Ker}(V_1) &= \begin{cases} G/\gamma_2(G), & \text{if } [\chi_2(G), \gamma_2(G)] = 1, \ k = 0, \ m \geq 3, \ w = 0, \ z = 0, \\ M_1/\gamma_2(G), & \text{if } [\chi_2(G), \gamma_2(G)] = 1, \ k = 0, \ m \geq 3, \ w = 1, \ z = 0, \\ M_2/\gamma_2(G), & \text{if } [\chi_2(G), \gamma_2(G)] = 1, \ k = 0, \ m \geq 4, \ w = 0, \ z > 0, \\ G/\gamma_2(G), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G), \ k = 1, \ m \geq 5, \ p \geq 3, \\ G/\gamma_2(G), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G), \ k \geq 2, \ m \geq 6, \ p \geq 5, \end{cases} \\ \text{Ker}(V_i) &= \begin{cases} M_1/\gamma_2(G), & \text{if } m = 3, \ w = 1, \ z = 0, \\ G/\gamma_2(G), & \text{if } m = 3, \ w = 0, \ z = 0, \\ G/\gamma_2(G), & \text{if } m \geq 4, \end{cases} \quad \text{for } 2 \leq i \leq p + 1. \end{aligned}$$

(2) *The singulets $\varkappa(i)$ of transfer types of G with $1 \leq i \leq p + 1$ are given by*

$$\begin{aligned} \varkappa(1) &= \begin{cases} 0, & \text{if } m \geq 3, \ G \simeq G_0^{(m)}(0, 0), \\ 1, & \text{if } m \geq 3, \ G = G_0^{(m)}(0, 1), \\ 2, & \text{if } m \geq 4, \ G = G_0^{(m)}(z, 0), \ z > 0, \\ 0, & \text{if } [\chi_2(G), \gamma_2(G)] > 1, \ k \geq 1, \ m \geq 5, \end{cases} \\ \varkappa(i) &= \begin{cases} 1, & \text{if } m = 3, \ G = G_0^{(3)}(0, 1), \\ 0, & \text{if } m = 3, \ G \simeq G_0^{(3)}(0, 0), \\ 0, & \text{if } m \geq 4, \end{cases} \quad \text{for } 2 \leq i \leq p + 1. \end{aligned}$$

(3) *The multiplet $\varkappa = (\varkappa(1), \dots, \varkappa(p + 1))$ of transfer types of G is given by*

$$\varkappa = \begin{cases} \overbrace{(1, \dots, 1)}^{p+1 \text{ times}}, \ \nu = 0, & \text{if } m = 3, \ G = G_0^{(3)}(0, 1), \\ \overbrace{(0, \dots, 0)}^{p+1 \text{ times}}, \ \nu = p + 1, & \text{if } m \geq 3, \ G \simeq G_0^{(m)}(0, 0), \\ \overbrace{(1, 0, \dots, 0)}^{p \text{ times}}, \ \nu = p, & \text{if } m \geq 4, \ G = G_0^{(m)}(0, 1), \\ \overbrace{(2, 0, \dots, 0)}^{p \text{ times}}, \ \nu = p, & \text{if } m \geq 4, \ G = G_0^{(m)}(z, 0), \ z > 0, \\ \overbrace{(0, \dots, 0)}^{p+1 \text{ times}}, \ \nu = p + 1, & \text{if } [\chi_2(G), \gamma_2(G)] > 1, \ k \geq 1, \ m \geq 5. \end{cases}$$

Proof. According to theorem 2.2, the images of the transfers in dependence on the invariants k and m , for elements $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $0 \leq j, \ell \leq p - 1$, are given by

$$\begin{aligned} V_1(x^j y^\ell \gamma_2(G)) &= \begin{cases} s_{m-1}^{wj+z\ell} \cdot 1, & \text{if } k = 0, \\ 1 \cdot \gamma_{m-k}(G), & \text{if } k \geq 1, \end{cases} \\ V_i(x^j y^\ell \gamma_2(G)) &= \begin{cases} s_2^{wj+z\ell} \cdot 1, & \text{if } m = 3, \\ 1 \cdot \gamma_3(G), & \text{if } m \geq 4, \end{cases} \quad \text{for } 2 \leq i \leq p + 1. \end{aligned}$$

To determine the transfer kernel $\text{Ker}(V_i)$ for $1 \leq i \leq p+1$, we have to solve the equation $V_i(x^j y^\ell \gamma_2(G)) = 1 \cdot \gamma_2(M_i)$ with respect to the element $x^j y^\ell$, that is, with respect to j, ℓ , using the image of the transfer V_i .

Since we have the trivial image $V_i(x^j y^\ell \gamma_2(G)) = 1 \cdot \gamma_3(G) = 1 \cdot \gamma_2(M_i)$ for $m \geq 4$ and $2 \leq i \leq p+1$, the exponents $0 \leq j, \ell \leq p-1$ can both be selected arbitrarily, and thus $\text{Ker}(V_i) = G/\gamma_2(G)$ is two-dimensional, that is $\varkappa(i) = 0$.

Similarly, we have the trivial image $V_1(x^j y^\ell \gamma_2(G)) = 1 \cdot \gamma_{m-k}(G) = 1 \cdot \gamma_2(M_1)$ for $k \geq 1$ and $i = 1$, and thus $\text{Ker}(V_1) = G/\gamma_2(G)$, that is $\varkappa(1) = 0$.

In all other cases, we need the concrete relational exponents w, z of the representative $G_a^{(m)}(z, w)$ of the isomorphism class of the given metabelian p -group G of maximal class, where a denotes the system $(a(m-k), \dots, a(m-1))$ of exponents in the general relation

$$[y, s_2] = \prod_{r=1}^k s_{m-r}^{a(m-r)} \in [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G)$$

by Miech [21, p.332, Th.2, (2)], which, however, does not enter the image of the transfer.

For $m = 3$, there are only the two isomorphism classes of the extra special p -groups $G_0^{(3)}(0, 0)$ of exponent p and $G_0^{(3)}(0, 1)$ of exponent p^2 , for which we have $k = 0$, and thus $a = 0$ means the empty family $(a(m-r))_{1 \leq r \leq k}$.

In the case $w = 0, z = 0$, we obtain the image $V_i(x^j y^\ell \gamma_2(G)) = s_2^{wj+z\ell} \cdot 1 = 1$ for $i = 1$ and also for $2 \leq i \leq p+1$, and thus $\text{Ker}(V_i) = G/\gamma_2(G)$, that is $\varkappa(i) = 0$.

In the case $w = 1, z = 0$, however, we have the image $V_i(x^j y^\ell \gamma_2(G)) = s_2^{wj+z\ell} \cdot 1 = s_2^j \cdot 1$ for $i = 1$ and also for $2 \leq i \leq p+1$. Therefore, we must have $j = 0$, whereas ℓ can be selected arbitrarily. Since $\langle y, \gamma_2(G) \rangle = M_1$, it follows that $\text{Ker}(V_i) = M_1/\gamma_2(G)$, that is $\varkappa(i) = 1$.

It only remains to investigate $i = 1$ for $m \geq 4$ and $k = 0$, that is, for metabelian p -groups G of maximal class with an abelian normal subgroup M_1 . According to Blackburn [6, p.88, Th.4.3], there exist two isomorphism classes $G_0^{(m)}(0, 0)$, $G_0^{(m)}(0, 1)$ with $z = 0$, and in general several isomorphism classes $G_0^{(m)}(z, 0)$ with $z > 0$, in this situation.

In the case $w = 0, z = 0$, we have $V_1(x^j y^\ell \gamma_2(G)) = s_{m-1}^{wj+z\ell} \cdot 1 = 1$ and thus $\text{Ker}(V_1) = G/\gamma_2(G)$, that is $\varkappa(1) = 0$.

In the case $w = 1, z = 0$, we obtain $V_1(x^j y^\ell \gamma_2(G)) = s_{m-1}^{wj+z\ell} \cdot 1 = s_{m-1}^j \cdot 1$. Thus, we must have $j = 0$, but ℓ remains arbitrary. Since $\langle y, \gamma_2(G) \rangle = M_1$, it follows that $\text{Ker}(V_1) = M_1/\gamma_2(G)$, that is $\varkappa(1) = 1$.

In the case $w = 0, z > 0$, we have $V_1(x^j y^\ell \gamma_2(G)) = s_{m-1}^{wj+z\ell} \cdot 1 = s_{m-1}^{z\ell} \cdot 1$. This enforces $\ell = 0$, whereas j can be selected arbitrarily. Since $\langle x, \gamma_2(G) \rangle = M_2$, it follows that $\text{Ker}(V_1) = M_2/\gamma_2(G)$, that is $\varkappa(1) = 2$.

According to [6, p.82], the possible maximum of the invariant k depends on m and p . \square

The irregular images of the transfers in the exceptional case $p = 2$ exert a decisive influence on the transfer kernels and cause considerable deviations of the multiplets of transfer types from the uniform standard case of odd primes $p \geq 3$.

Theorem 2.6. *Let $G = \langle x, y \rangle$ be a metabelian 2-group of maximal class and of order $|G| = 2^m$, where $m \geq 3$. Suppose that the generators are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, $y \in \chi_2(G) \setminus \gamma_2(G)$, and the relations (1) are satisfied with exponents $0 \leq w, z \leq 1$. Let the maximal normal subgroups M_1, \dots, M_3 of G be ordered by $M_1 = \langle y, \gamma_2(G) \rangle$, $M_2 = \langle x, \gamma_2(G) \rangle$, and $M_3 = \langle xy, \gamma_2(G) \rangle$. Then $M_1 = \chi_2(G)$, if $m \geq 4$, and the following statements hold.*

- (1) *The kernels of the transfers $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ for $1 \leq i \leq 3$ are given by*

$$\begin{aligned}
\text{Ker}(V_1) &= \begin{cases} G/\gamma_2(G), & \text{if } m \geq 3, w = 0, z = 0, \\ M_1/\gamma_2(G), & \text{if } m \geq 3, w = 1, z = 0, \\ M_2/\gamma_2(G), & \text{if } m \geq 4, w = 0, z = 1, \end{cases} \\
\text{Ker}(V_2) &= \begin{cases} M_2/\gamma_2(G), & \text{if } m = 3, w = 1, z = 0, \\ M_3/\gamma_2(G), & \text{otherwise,} \end{cases} \\
\text{Ker}(V_3) &= \begin{cases} M_3/\gamma_2(G), & \text{if } m = 3, w = 1, z = 0, \\ M_2/\gamma_2(G), & \text{otherwise.} \end{cases}
\end{aligned}$$

(2) The singulets $\varkappa(i)$ of transfer types of G with $1 \leq i \leq 3$ are given by

$$\begin{aligned}
\varkappa(1) &= \begin{cases} 0, & \text{if } m \geq 3, G = G_0^{(m)}(0, 0) = D(2^m), \\ 1, & \text{if } m \geq 3, G = G_0^{(m)}(0, 1) = Q(2^m), \\ 2, & \text{if } m \geq 4, G = G_0^{(m)}(1, 0) = S(2^m), \end{cases} \\
\varkappa(2) &= \begin{cases} 2, & \text{if } m = 3, G = G_0^{(3)}(0, 1) = Q(8), \\ 3, & \text{otherwise,} \end{cases} \\
\varkappa(3) &= \begin{cases} 3, & \text{if } m = 3, G = G_0^{(3)}(0, 1) = Q(8), \\ 2, & \text{otherwise.} \end{cases}
\end{aligned}$$

(3) The multiplet $\varkappa = (\varkappa(1), \varkappa(2), \varkappa(3))$ of transfer types of G is given by

$$\varkappa = \begin{cases} (1, 2, 3), \nu = 0, & \text{if } m = 3, G = G_0^{(3)}(0, 1) = Q(8), \\ (0, 3, 2), \nu = 1, & \text{if } m \geq 3, G = G_0^{(m)}(0, 0) = D(2^m), \\ (1, 3, 2), \nu = 0, & \text{if } m \geq 4, G = G_0^{(m)}(0, 1) = Q(2^m), \\ (2, 3, 2), \nu = 0, & \text{if } m \geq 4, G = G_0^{(m)}(1, 0) = S(2^m). \end{cases}$$

Definition 2.3. We call the multiplets of transfer types \varkappa in the preceding theorems 2.5 and 2.6 *canonical*, since they only appear in this form, if the group generators x, y are selected as described in the assumptions and the maximal normal subgroups are arranged in the defined order.

Proof. By theorem 2.3, the images of the transfers for elements of the shape $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$, $0 \leq j, \ell \leq 1$, and in dependence on the invariant m , are given by

$$\begin{aligned}
V_1(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} \cdot 1, \\
V_2(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} s_2^{-(j+\ell)} \gamma_3(G) = \begin{cases} s_2^{(w-1)j+(z-1)\ell} \cdot 1, & \text{if } m = 3, \\ s_2^{-(j+\ell)} \gamma_3(G), & \text{if } m \geq 4, \end{cases} \\
V_3(x^j y^\ell \gamma_2(G)) &= s_{m-1}^{wj+z\ell} s_2^{-\ell} \gamma_3(G) = \begin{cases} s_2^{wj+(z-1)\ell} \cdot 1, & \text{if } m = 3, \\ s_2^{-\ell} \gamma_3(G), & \text{if } m \geq 4. \end{cases}
\end{aligned}$$

To determine the transfer kernel $\text{Ker}(V_i)$ for $1 \leq i \leq 3$, we have to solve the equation $V_i(x^j y^\ell \gamma_2(G)) = 1 \cdot \gamma_2(M_i)$ with respect to the element $x^j y^\ell$, that is, with respect to j, ℓ , for the corresponding image of the transfer V_i .

Since the image of the first transfer V_1 is regular, we can use the result for $\text{Ker}(V_1)$ of theorem 2.5 with $k = 0$.

The images of the second and third transfer V_2, V_3 are independent from the relational exponents w, z in the case $m \geq 4$, that is, they are equal for all groups with index of nilpotency $m \geq 4$. The equation $V_2(x^j y^\ell \gamma_2(G)) = s_2^{-(j+\ell)} \gamma_3(G) = 1 \cdot \gamma_3(G)$ enforces $j + \ell = 0$, that is $j = \ell$, because we are dealing with 2-groups, and $\langle xy, \gamma_2(G) \rangle = M_3$ implies $\text{Ker}(V_2) = M_3/\gamma_2(G)$, $\varkappa(2) = 3$.

The equation $V_3(x^j y^\ell \gamma_2(G)) = s_2^{-\ell} \gamma_3(G) = 1 \cdot \gamma_3(G)$ implies $\ell = 0$, whereas j remains arbitrary. Since $\langle x, \gamma_2(G) \rangle = M_2$, we obtain $\text{Ker}(V_3) = M_2/\gamma_2(G)$, $\kappa(3) = 2$.

For $m = 3$, we have only two isomorphism classes of metabelian 2-groups of maximal class, $G_0^{(3)}(0, 0) = D(8)$ and $G_0^{(3)}(0, 1) = Q(8)$.

In the case $w = 0, z = 0$, we obtain

$$\begin{aligned} V_2(x^j y^\ell \gamma_2(G)) &= s_2^{(w-1)j+(z-1)\ell} \cdot 1 = s_2^{-(j+\ell)} \text{ and} \\ V_3(x^j y^\ell \gamma_2(G)) &= s_2^{wj+(z-1)\ell} \cdot 1 = s_2^{-\ell} \text{ and thus, similarly as for } m \geq 4, \\ \text{Ker}(V_2) &= M_3/\gamma_2(G), \kappa(2) = 3 \text{ and } \text{Ker}(V_3) = M_2/\gamma_2(G), \kappa(3) = 2. \end{aligned}$$

In the case $w = 1, z = 0$, however, we have

$$\begin{aligned} V_2(x^j y^\ell \gamma_2(G)) &= s_2^{(w-1)j+(z-1)\ell} \cdot 1 = s_2^{-\ell} \text{ and} \\ V_3(x^j y^\ell \gamma_2(G)) &= s_2^{wj+(z-1)\ell} \cdot 1 = s_2^{j-\ell} \text{ and thus, conversely,} \\ \text{Ker}(V_2) &= M_2/\gamma_2(G), \kappa(2) = 2 \text{ and } \text{Ker}(V_3) = M_3/\gamma_2(G), \kappa(3) = 3. \end{aligned}$$

□

In table 1 we compare the multiplets of transfer types of metabelian p -groups of maximal class for $p = 2$ and $p \geq 3$. Here, κ denotes the multiplet of coarse transfer types expressed with the aid of condition (B) by Taussky [26, p.435], which is given by Kisilevsky [16, p.273, Th.2] and by Benjamin and Snyder [4, p.163, §2]. Further, $1 \leq z \leq p-1$ is one of the exponents in the relations (1). Examples of number fields with these principalisation types are given in [19, 20].

TABLE 1. Transfer types of corresponding p -groups for $p = 2$ and $p \geq 3$

$p = 2$				$p \geq 3$	
κ	\varkappa	2-group	m	\varkappa	p -group
(000)	(000)	$C(2) \times C(2)$	2	$\overbrace{(0, \dots, 0)}^{p+1 \text{ times}}$	$C(p) \times C(p)$
(123)	(123)	$G_0^{(3)}(0, 1) = Q(8)$	3	$\overbrace{(1, \dots, 1)}^{p+1 \text{ times}}$	$G_0^{(3)}(0, 1)$
(0BB)	(032)	$G_0^{(m)}(0, 0) = D(2^m)$	≥ 3	$\overbrace{(0, \dots, 0)}^{p+1 \text{ times}}$	$G_0^{(m)}(0, 0)$
(1BB)	(132)	$G_0^{(m)}(0, 1) = Q(2^m)$	≥ 4	$\overbrace{(1, 0, \dots, 0)}^{p \text{ times}}$	$G_0^{(m)}(0, 1)$
(BBB)	(232)	$G_0^{(m)}(1, 0) = S(2^m)$	≥ 4	$\overbrace{(2, 0, \dots, 0)}^{p \text{ times}}$	$G_0^{(m)}(z, 0)$

2.5. Combinatorially possible transfer types of 2-groups. In this subsection, we arrange the combinatorially possible S_3 -orbits of the 4^3 triplets $\varkappa \in [0, 3]^3$ according to increasing cardinality of the image and decreasing number of fixed points. Table 2 shows the partial triplets and table 3 the total triplets as the possible transfer types of metabelian 2-groups G with $G/\gamma_2(G)$ of type $(2, 2)$, resp. principalisation types of base fields K with 2-class group $\text{Cl}_2(K)$ of type $(2, 2)$. The orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\varkappa) = (|\varkappa^{-1}\{i\}|)_{0 \leq i \leq 3}$ the family of occupation numbers of the selected orbit representative \varkappa and by $F = \{1 \leq i \leq 3 \mid \varkappa(i) = i\}$ the set of fixed points of \varkappa . In the characterising property, needed for equal numbers of fixed points, let $I = \{\varkappa(i) \mid 1 \leq i \leq 3\}$ be the image of \varkappa and $Z = \varkappa^{-1}\{0\}$ the preimage of zero under \varkappa .

If an orbit can be realised as a transfer type, then a suitable 2-group G is given, according to theorem 2.6.

TABLE 2. The seven S_3 -orbits of triplets $\varkappa \in [1, 3]^3$ with $\nu = 0$

Sec.	Nr.	repres. of orbit \varkappa	occupation numbers $o(\varkappa)$	fixed points $ F $	charact. property	cardinality of orbit $ \varkappa^{S_3} $	realising 2-group G
A	1	(111)	(0300)	1	constant	3	impossible
B	2	(121)	(0210)	2	almost	6	impossible
B	3	(112)	(0210)	1	con-	6	impossible
S	4	(211)	(0210)	0	stant	6	$G_0^{(m)}(1, 0) = S(2^m)$, $m \geq 4$
Q	5	(123)	(0111)	3	identity	1	$G_0^{(3)}(0, 1) = Q(8)$, $m = 3$
Q	6	(132)	(0111)	1	transposition	3	$G_0^{(m)}(0, 1) = Q(2^m)$, $m \geq 4$
C	7	(231)	(0111)	0	3-cycle	2	impossible
Total number:						27	

TABLE 3. The nine S_3 -orbits of triplets $\varkappa \in [0, 3]^3 \setminus [1, 3]^3$ with $1 \leq \nu \leq 3$

Sec.	Nr.	repres. of orbit \varkappa	occupation numbers $o(\varkappa)$	fixed points $ F $	charact. property	cardinality of orbit $ \varkappa^{S_3} $	realising 2-group G
a	1	(000)	(3000)	0	constant	1	$C(2) \times C(2)$, $m = 2$
b	2	(100)	(2100)	1		3	impossible
b	3	(010)	(2100)	0		6	impossible
c	4	(110)	(1200)	1		6	impossible
c	5	(011)	(1200)	0		3	impossible
e	6	(120)	(1110)	2	identity with 0	3	impossible
e	7	(021)	(1110)	1		6	impossible
d	8	(210)	(1110)	0	$Z \not\subset I$	3	$G_0^{(m)}(0, 0) = D(2^m)$, $m \geq 3$
e	9	(012)	(1110)	0	$Z \subset I$	6	impossible
Total number:						$37 = 64 - 27$	

S.4 is the transfer type of the semi-dihedral group $S(2^m) = G_0^{(m)}(1, 0)$ with $m \geq 4$, Q.5 the transfer type of the quaternion group $Q(8) = G_0^{(3)}(0, 1)$, $m = 3$, Q.6 the transfer type of the generalised quaternion group $Q(2^m) = G_0^{(m)}(0, 1)$, $m \geq 4$, a.1 the transfer type of the elementary abelian bicyclic 2-group of type $(2, 2)$, $m = 2$, and, finally, d.8 is the transfer type of the dihedral group $D(2^m) = G_0^{(m)}(0, 0)$ with $m \geq 3$.

3. TRANSFERS OF A METABELIAN 3-GROUP OF NON-MAXIMAL CLASS

The transfer kernels $\text{Ker}(V_i)$ and transfer types $\varkappa(G)$ of metabelian 3-groups G of non-maximal class with abelianisation $G/\gamma_2(G)$ of type $(3, 3)$ have been determined in Nebelung's thesis [23]. These transfer types are summarised in the tables 6 and 7 of subsection 3.3.

However, our concrete numerical investigation [19] of all 4 596 quadratic base fields $K = \mathbb{Q}(\sqrt{D})$ with 3-class group of type $(3, 3)$ and discriminant in the range $-10^6 < D < 10^7$ has revealed that supplementary criteria are necessary to distinguish between two kinds of the transfer types d.19, d.23, and d.25 of the second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of arbitrary base fields K . To be able to describe these two kinds of transfer types of section d, we must recall some results of [23].

The metabelian 3-groups G of non-maximal class, that is, of coclass $\text{cc}(G) \geq 2$, cannot be CF-groups, since they must have at least one bicyclic factor $\gamma_3(G)/\gamma_4(G)$. Similarly as in section 2 of [19], we declare an isomorphism invariant $e = e(G)$ of G by $e + 1 = \min\{3 \leq j \leq m \mid 1 \leq |\gamma_j(G)/\gamma_{j+1}(G)| \leq 3\}$. This invariant $2 \leq e \leq m - 1$ characterises the first cyclic factor $\gamma_{e+1}(G)/\gamma_{e+2}(G)$ of the lower central series of G , except $\gamma_2(G)/\gamma_3(G)$, which is always cyclic. e can be calculated from the 3-exponent n of the order $|G| = 3^n$ and the class $\text{cl}(G) = m - 1$, resp. the index m , of nilpotency of G by the formula $e = n - m + 2$. Since the coclass of G is given by $\text{cc}(G) = n - \text{cl}(G) = n - m + 1$, we have the relation $e = \text{cc}(G) + 1$.

The isomorphism classes of all metabelian 3-groups G with abelianisation $G/\gamma_2(G)$ of type $(3, 3)$ can be represented as nodes of a directed tree with root $C(3) \times C(3)$ [23, p.181 ff]. The tree contains infinite chains, all of whose nodes have the same transfer type. There exists a single chain of transfer type a.1, giving rise to all groups of maximal class [6] with invariant $e = 2$. This is the main line of the unique coclass tree in the coclass graph $\mathcal{G}(3, 1)$ [3, 17, 9, 8]. Further there are three chains of the transfer types b.10, c.21, and c.18, giving rise to all groups of second maximal class [3] with invariant $e = 3$. These are main lines of coclass trees in coclass graph $\mathcal{G}(3, 2)$. Eventually, for each $r \geq 3$, there are finitely many chains of the transfer types b.10, d.23, d.25, and d.19, giving rise to groups of lower than second maximal class with invariant $e = r + 1$. These are main lines of coclass trees in coclass graph $\mathcal{G}(3, r)$ with a certain periodicity of length two with respect to r .

Only the transfer types d.19, d.23, and d.25 can occur for both, *internal* nodes on the corresponding chains with $e \geq 4$ and *terminal* successors of nodes on all chains with transfer type b.10. We will show that the distinction of internal and terminal nodes is generally possible with the aid of the canonical multiplet \varkappa of transfer types of G for arbitrary base fields K , and in particular by means of the parity of the index m of nilpotency of G for quadratic base fields $K = \mathbb{Q}(\sqrt{D})$.

3.1. Images of the transfers. For a group G of non-maximal class we need a generalisation of the group $\chi_2(G)$. Denoting by m the index of nilpotency of G , we let $\chi_j(G)$ with $2 \leq j \leq m - 1$ be the centralisers of two-step factor groups $\gamma_j(G)/\gamma_{j+2}(G)$ of the lower central series, that is, the biggest subgroups of G with the property $[\chi_j(G), \gamma_j(G)] \leq \gamma_{j+2}(G)$. They form an ascending chain of characteristic subgroups of G , $\gamma_2(G) \leq \chi_2(G) \leq \dots \leq \chi_{m-2}(G) < \chi_{m-1}(G) = G$, which contain the commutator group $\gamma_2(G)$. $\chi_j(G)$ coincides with G , if and only if $j \geq m - 1$. Similarly as in section 2 of [19], we characterise the smallest two-step centraliser different from the commutator group by an isomorphism invariant $s = s(G) = \min\{2 \leq j \leq m - 1 \mid \chi_j(G) > \gamma_2(G)\}$.

The assumptions of the following theorem 3.1 for a metabelian 3-group G of non-maximal class with abelianisation $G/\gamma_2(G)$ of type $(3, 3)$ can always be satisfied, according to [23].

Theorem 3.1. *Let G be a metabelian 3-group of non-maximal class with abelianisation $G/\gamma_2(G)$ of type $(3, 3)$. Assume that G has order $|G| = 3^n$, class $\text{cl}(G) = m - 1$, and invariant $e = n - m + 2 \geq 3$, where $4 \leq m < n \leq 2m - 3$. Let generators of $G = \langle x, y \rangle$ be selected such that $\gamma_3(G) = \langle y^3, x^3, \gamma_4(G) \rangle$, $x \in G \setminus \chi_s(G)$, if $s < m - 1$, and $y \in \chi_s(G) \setminus \gamma_2(G)$. Suppose that the order of the four maximal normal subgroups of G is defined by $M_i = \langle g_i, \gamma_2(G) \rangle$ with $g_1 = y$, $g_2 = x$, $g_3 = xy$, and $g_4 = xy^{-1}$. Let the main commutator of G be declared by $s_2 = t_2 = [y, x] \in \gamma_2(G)$ and higher commutators recursively by $s_j = [s_{j-1}, x]$, $t_j = [t_{j-1}, y] \in \gamma_j(G)$ for $j \geq 3$. Starting with the powers $\sigma_3 = y^3$, $\tau_3 = x^3 \in \gamma_3(G)$, let $\sigma_j = [\sigma_{j-1}, x]$, $\tau_j = [\tau_{j-1}, y] \in \gamma_j(G)$ for $j \geq 4$. With exponents $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$, let the following relations be satisfied*

$$(5) \quad s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1}, \quad s_3 \sigma_3 \sigma_4 = \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma} \tau_e^{\delta}, \quad t_3^{-1} \tau_3 \tau_4 = \sigma_{m-2}^{\rho\delta} \sigma_{m-1}^{\alpha} \tau_e^{\beta}, \quad \tau_{e+1} = \sigma_{m-1}^{-\rho}.$$

Finally, let $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ with $0 \leq k \leq 1$.

If the cosets are represented in the form $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $-1 \leq j, \ell \leq 1$, then the images of the transfers $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ are given by

$$\begin{aligned}
V_1(g\gamma_2(G)) &= \begin{cases} \sigma_{m-1}^{\gamma\ell} \tau_e^{\delta\ell} \tau_3^j \gamma_2(M_1), & \text{if } [\chi_s(G), \gamma_e(G)] = 1, \ k = 0, \\ \sigma_{m-2}^{\rho\beta\ell} \sigma_{m-1}^{(\gamma-\rho\beta)\ell} \tau_e^{\delta\ell} \tau_3^j \gamma_2(M_1), & \text{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), \ k = 1, \ m \geq 5, \end{cases} \\
V_2(g\gamma_2(G)) &= \begin{cases} \sigma_3^\ell \sigma_{m-1}^{\alpha j} \tau_e^{\beta j} \gamma_2(M_2), & \text{if } [\chi_s(G), \gamma_e(G)] = 1, \ k = 0, \\ \sigma_3^\ell \sigma_{m-2}^{\rho\delta j} \sigma_{m-1}^{(\alpha+\rho\beta)j} \tau_e^{\beta j} \gamma_2(M_2), & \text{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), \ k = 1, \ m \geq 5, \end{cases} \\
V_i(g\gamma_2(G)) &= \sigma_3^\ell \tau_3^j \gamma_2(M_i) \text{ for } 3 \leq i \leq 4.
\end{aligned}$$

Proof. First we prove a formula for the third trace element acting as symbolic exponent. Let G be an arbitrary group with a normal subgroup $N < G$. For $u \in N$ and $h \in G \setminus N$ we have

$$(6) \quad u^{S_3(h)} \equiv u^3[u, h]^3[[u, h], h] \pmod{\gamma_2(N)}.$$

This follows from $u^{S_3(h)} = u^{1+h+h^2} = u \cdot h^{-1}uh \cdot h^{-2}uh^2$ and
 $u^3[u, h]^3[[u, h], h] = u^3[u, h]^3[u, h]^{-1}h^{-1}[u, h]h$
 $= u^3[u, h]^2h^{-1} \cdot u^{-1}h^{-1}uh \cdot h = u^3 \cdot u^{-1}h^{-1}uh \cdot u^{-1}h^{-1}uh \cdot h^{-1} \cdot u^{-1}h^{-1}uh^2$
 $= u^2 \cdot h^{-1}uh u^{-1} \cdot h^{-2}uh^2 = u^2 \cdot u^{-1}h^{-1}uh[h^{-1}uh, u^{-1}] \cdot h^{-2}uh^2$
 $\equiv u \cdot h^{-1}uh \cdot h^{-2}uh^2 \pmod{\gamma_2(N)}$, since $h^{-1}uh, u^{-1} \in N$, and thus $[h^{-1}uh, u^{-1}] \in [N, N] = \gamma_2(N)$.

For the first transfer V_1 we have $x \in G \setminus M_1$ and $y \in M_1$, and thus
 $V_1(x\gamma_2(G)) = x^3\gamma_2(M_1) = \tau_3\gamma_2(M_1)$ for the generator x .

Now we use formula(6) and the relations (5) for the generator y ,

$$\begin{aligned}
V_1(y\gamma_2(G)) &= y^{S_3(x)}\gamma_2(M_1) = y^3[y, x]^3[[y, x], x]\gamma_2(M_1) = y^3s_2^3s_3\gamma_2(M_1) \\
&= \sigma_3 \cdot \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1} \cdot \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^\gamma \tau_e^\delta \gamma_2(M_1) = \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma-\rho\beta} \tau_e^\delta \gamma_2(M_1), \\
&\text{since } \tau_4 \in \gamma_2(M_1) = \langle \tau_3, \tau_4, \dots, \tau_{e+1} \rangle, \text{ by [19, Cor.4.1.1]}.
\end{aligned}$$

For the second transfer V_2 we have $x \in M_2$ and $y \in G \setminus M_2$, and thus
 $V_2(y\gamma_2(G)) = y^3\gamma_2(M_2) = \sigma_3\gamma_2(M_2)$ for the generator y .

Now we use formula (6) and the relations (5) for the generator x ,

$$\begin{aligned}
V_2(x\gamma_2(G)) &= x^{S_3(y)}\gamma_2(M_2) = x^3[x, y]^3[[x, y], y]\gamma_2(M_2) = x^3s_2^{-3}t_3^{-1}\gamma_2(M_2) \\
&= \tau_3 \cdot \sigma_4^{-1} \sigma_{m-1}^{\rho\beta} \tau_4 \cdot \tau_3^{-1} \tau_4^{-1} \sigma_{m-2}^{\rho\delta} \sigma_{m-1}^\alpha \tau_e^\beta \gamma_2(M_2) = \sigma_{m-2}^{\rho\delta} \sigma_{m-1}^{\alpha+\rho\beta} \tau_e^\beta \gamma_2(M_2),
\end{aligned}$$

since $[[x, y], y] = [s_2^{-1}, y] = [s_2, y]^{-s_2^{-1}} = (t_3^{-1})^{s_2^{-1}} = t_3^{-1}$, by the formula for inverses,
and $\sigma_4 \in \gamma_2(M_2) = \langle s_3, \sigma_4, \dots, \sigma_{m-1} \rangle$, by [19, Cor.4.1.1].

Thus, the images for an arbitrary coset $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $-1 \leq j, \ell \leq 1$ are

$$\begin{aligned}
V_1(g\gamma_2(G)) &= \tau_3^j \sigma_{m-2}^{\rho\beta\ell} \sigma_{m-1}^{(\gamma-\rho\beta)\ell} \tau_e^{\delta\ell} \gamma_2(M_1), \\
V_2(g\gamma_2(G)) &= \sigma_{m-2}^{\rho\delta j} \sigma_{m-1}^{(\alpha+\rho\beta)j} \tau_e^{\beta j} \sigma_3^\ell \gamma_2(M_2).
\end{aligned}$$

The simplified images for $k = 0$ are a consequence of the central relation $[y, \tau_e] = \tau_{e+1}^{-1} = \sigma_{m-1}^\rho$ in $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G) \leq \zeta_1(G)$, since $k = 0$ is equivalent with $\rho = 0$.

For the other two transfers V_i with $3 \leq i \leq 4$, we have $x \in G \setminus M_i$ and $y \in G \setminus M_i$, and thus

$$\begin{aligned}
V_i(x\gamma_2(G)) &= x^3\gamma_2(M_i) = \tau_3\gamma_2(M_i), \\
V_i(y\gamma_2(G)) &= y^3\gamma_2(M_i) = \sigma_3\gamma_2(M_i), \text{ and therefore } V_i(g\gamma_2(G)) = \tau_3^j \sigma_3^\ell \gamma_2(M_i).
\end{aligned}$$

According to [23], the possible maximum of invariant k depends on the index of nilpotency m . \square

3.2. Kernels of the transfers. Since we are particularly interested in second 3-class groups G of quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ with transfer types in section d, we now focus on metabelian 3-groups G of coclass $\text{cc}(G) \geq 3$.

Theorem 3.2. *Let G be a metabelian 3-group, having abelianisation $G/\gamma_2(G)$ of type $(3, 3)$, of lower than second maximal class, $e \geq 4$, and thus with index of nilpotency $m \geq 6$ or $m = 5, k = 0$. For the generators of $G = \langle x, y \rangle$ and the order of the maximal normal subgroups M_1, \dots, M_4 let the assumptions of theorem 3.1 be satisfied. Suppose that G is defined by the relations (5) with a system of exponents $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$ and that $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ with $0 \leq k \leq 1$.*

Finally, let a function f of two parameters $-1 \leq \lambda, \mu \leq 1$ be defined by

$$f(\lambda, \mu) = \begin{cases} 0, & \text{if } \lambda = 0, \mu = 0, \\ 1, & \text{if } \lambda = \pm 1, \mu = 0, \\ 2, & \text{if } \lambda = 0, \mu = \pm 1, \\ 3, & \text{if } \lambda = \pm 1, \mu = \pm 1, \lambda = -\mu, \\ 4, & \text{if } \lambda = \pm 1, \mu = \pm 1, \lambda = \mu. \end{cases}$$

Then the singulets of transfer types of G are given by

$$\begin{aligned} \varkappa(1) &= \begin{cases} f(\alpha, \gamma), & \text{if } k = 0, \\ f(\delta, \beta), & \text{if } k = 1, \end{cases} \\ \varkappa(2) &= f(\beta, \delta), \\ \varkappa(3) &= 4, \\ \varkappa(4) &= 3. \end{aligned}$$

Therefore, the kernels of the transfers $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ with $1 \leq i \leq 4$ are given by $\text{Ker}(V_i) = M_{\varkappa(i)}/\gamma_2(G)$, if $1 \leq \varkappa(i) \leq 4$, and $\text{Ker}(V_i) = G/\gamma_2(G)$, if $\varkappa(i) = 0$.

Proof. We start with Corollary 4.1.1 of our paper [19], where the commutator groups $\gamma_2(M_i)$ of the maximal normal subgroups M_i of G are determined:

$$\begin{aligned} \gamma_2(M_1) &= \langle t_3, \tau_4, \dots, \tau_{e+1} \rangle, \\ \gamma_2(M_2) &= \langle s_3, \sigma_4, \dots, \sigma_{m-1} \rangle, \\ \gamma_2(M_3) &= \langle s_3 t_3, \gamma_4(G) \rangle, \\ \gamma_2(M_4) &= \langle s_3 t_3^{-1}, \gamma_4(G) \rangle, \end{aligned}$$

where $\gamma_4(G) = \langle \sigma_4, \dots, \sigma_{m-1}, \tau_4, \dots, \tau_{e+1} \rangle$.

As a consequence of the relations (5), we have

$$\begin{aligned} s_3 &= \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma} \tau_e^{\delta}, \\ t_3 &= \tau_3 \tau_4 \sigma_{m-2}^{-\rho\delta} \sigma_{m-1}^{-\alpha} \tau_e^{-\beta}, \\ s_3 t_3 &= \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho(\beta-\delta)} \sigma_{m-1}^{\gamma-\alpha} \tau_3 \tau_4 \tau_e^{\delta-\beta}, \\ s_3 t_3^{-1} &= \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho(\beta+\delta)} \sigma_{m-1}^{\alpha+\gamma} \tau_3^{-1} \tau_4^{-1} \tau_e^{\beta+\delta}. \end{aligned}$$

To determine the kernel $\text{Ker}(V_i)$ of the transfer $V_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$ for $1 \leq i \leq 4$, where cosets $g\gamma_2(G) \in G/\gamma_2(G)$ are represented by the generators x, y in the shape $g \equiv x^j y^\ell \pmod{\gamma_2(G)}$ with $-1 \leq j, \ell \leq 1$, the equation $V_i(g\gamma_2(G)) = V_i(x^j y^\ell \gamma_2(G)) = 1 \cdot \gamma_2(M_i)$ must be solved with respect to j, ℓ . For this purpose, we use the images of the transfers in theorem 3.1.

For the kernel of the first transfer we have $V_1(x^j y^\ell \gamma_2(G)) = \sigma_{m-2}^{\rho\beta\ell} \sigma_{m-1}^{(\gamma-\rho\beta)\ell} \tau_e^{\delta\ell} \tau_3^j \gamma_2(M_1) = 1 \cdot \gamma_2(M_1)$,

and thus $(\sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma-\rho\beta} \tau_e^{\delta})^\ell \tau_3^j \in \gamma_2(M_1) = \langle t_3, \tau_4, \dots, \tau_{e+1} \rangle = \langle \tau_3 \tau_4 \sigma_{m-2}^{-\rho\delta} \sigma_{m-1}^{-\alpha} \tau_e^{-\beta}, \tau_4, \dots, \tau_{e+1} \rangle$.

For $e \geq 4$ we obtain a simplification

$$(\sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma-\rho\beta})^\ell \tau_3^j \in \langle \tau_3 \sigma_{m-2}^{-\rho\delta} \sigma_{m-1}^{-\alpha}, \tau_4, \dots, \tau_{e+1} \rangle,$$

and thus $(\sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma-\rho\beta})^\ell \tau_3^j = (\sigma_{m-2}^{-\rho\delta} \sigma_{m-1}^{-\alpha} \tau_3)^r$ with $-1 \leq r \leq 1$.

Now we distinguish the values of the invariant $0 \leq k \leq 1$.

In the case $k = 0$, that is, for $\rho = 0$, we have $\sigma_{m-1}^{\gamma\ell} \tau_3^j = (\sigma_{m-1}^{-\alpha} \tau_3)^r$, therefore $j = r$, $\gamma\ell = -\alpha r$, and consequently $\alpha j = -\gamma\ell$.

In the case $k = 1$, however, that is, for $\rho = \pm 1$, we have $\sigma_{m-1} = \tau_{e+1}^{-\rho}$ with $e + 1 \geq 5$ and thus $\sigma_{m-2}^{\rho\beta\ell}\tau_3^j = \left(\sigma_{m-2}^{-\rho\delta}\tau_3\right)^r$, therefore $j = r$, $\rho\beta\ell = -\rho\delta r$, and consequently $\delta j = -\beta\ell$.

For the kernel of the second transfer we have $V_2(x^j y^\ell \gamma_2(G)) = \sigma_3^\ell \sigma_{m-2}^{\rho\delta j} \sigma_{m-1}^{(\alpha+\rho\beta)j} \tau_e^{\beta j} \gamma_2(M_2) = 1 \cdot \gamma_2(M_2)$, and thus $\sigma_3^\ell \left(\sigma_{m-2}^{\rho\delta} \sigma_{m-1}^{\alpha+\rho\beta} \tau_e^\beta\right)^j \in \gamma_2(M_2) = \langle s_3, \sigma_4, \dots, \sigma_{m-1} \rangle = \langle \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^\gamma \tau_e^\delta, \sigma_4, \dots, \sigma_{m-1} \rangle$. For $m \geq 6$ or $m = 5$, $\rho = 0$ we obtain the simplification $\sigma_3^\ell \tau_e^{\beta j} \in \langle \sigma_3^{-1} \tau_e^\delta, \sigma_4, \dots, \sigma_{m-1} \rangle$, and thus $\sigma_3^\ell \tau_e^{\beta j} = (\sigma_3^{-1} \tau_e^\delta)^r$ with $-1 \leq r \leq 1$, independently from α, γ, ρ . Consequently $\ell = -r$, $\beta j = \delta r$ and thus $\beta j = -\delta\ell$.

The solution of the equation $\lambda j = -\mu\ell$ for $1 \leq i \leq 2$, which depends on the relational exponents $(\lambda, \mu) \in \{(\alpha, \gamma), (\delta, \beta), (\beta, \delta)\}$, is obtained by the following distinction of cases.

In the case $\lambda = 0$, $\mu = 0$, we can select j, ℓ arbitrarily, and the kernel is $G/\gamma_2(G)$, $\kappa(i) = 0$.

In the case $\lambda = \pm 1$, $\mu = 0$ we must have $j = 0$ but ℓ remains arbitrary. Thus, the kernel equals $M_1/\gamma_2(G)$, $\kappa(i) = 1$.

In the case $\lambda = 0$, $\mu = \pm 1$, we must have $\ell = 0$ but j remains arbitrary. Thus, the kernel equals $M_2/\gamma_2(G)$, $\kappa(i) = 2$.

In the case $\lambda = \pm 1$, $\mu = \pm 1$, finally, we obtain that $\ell = j$ for $\lambda = -\mu$ and the kernel is $M_3/\gamma_2(G)$, $\kappa(i) = 3$. For $\lambda = \mu$, however, we must have $\ell = -j$ and the kernel equals $M_4/\gamma_2(G)$, $\kappa(i) = 4$.

For the kernel of the third transfer, we have $V_3(x^j y^\ell \gamma_2(G)) = \sigma_3^\ell \tau_3^j \gamma_2(M_3) = 1 \cdot \gamma_2(M_3)$ and $\sigma_3^\ell \tau_3^j \in \gamma_2(M_3) = \langle s_3 t_3, \gamma_4(G) \rangle = \langle \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho(\beta-\delta)} \sigma_{m-1}^{\gamma-\alpha} \tau_3 \tau_4 \tau_e^{\delta-\beta}, \sigma_4, \dots, \sigma_{m-1}, \tau_4, \dots, \tau_{e+1} \rangle$. In the case of a group of lower than second maximal class with $e \geq 4$, and thus either $m \geq 6$ or $m = 5$, $\rho = 0$, this relation becomes simply $\sigma_3^\ell \tau_3^j \in \langle \sigma_3^{-1} \tau_3, \gamma_4(G) \rangle$, and thus $\sigma_3^\ell \tau_3^j = (\sigma_3^{-1} \tau_3)^r$ with $-1 \leq r \leq 1$.

This means $\ell = -r$, $j = r$ and thus

$$\ell = -j, g \equiv (xy^{-1})^j \pmod{\gamma_2(G)}, g \in \langle xy^{-1}, \gamma_2(G) \rangle = M_4, \text{ that is, } \kappa(3) = 4.$$

For the kernel of the fourth transfer, we have $V_4(x^j y^\ell \gamma_2(G)) = \sigma_3^\ell \tau_3^j \gamma_2(M_4) = 1 \cdot \gamma_2(M_4)$ and $\sigma_3^\ell \tau_3^j \in \gamma_2(M_4) = \langle s_3 t_3^{-1}, \gamma_4(G) \rangle = \langle \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho(\beta+\delta)} \sigma_{m-1}^{\alpha+\gamma} \tau_3^{-1} \tau_4^{-1} \tau_e^{\beta+\delta}, \sigma_4, \dots, \sigma_{m-1}, \tau_4, \dots, \tau_{e+1} \rangle$. For $e \geq 4$, and thus either $m \geq 6$ or $m = 5$, $\rho = 0$, this relation becomes simply $\sigma_3^\ell \tau_3^j \in \langle \sigma_3^{-1} \tau_3^{-1}, \gamma_4(G) \rangle$, and thus $\sigma_3^\ell \tau_3^j = (\sigma_3^{-1} \tau_3^{-1})^r$ with $-1 \leq r \leq 1$.

Consequently $\ell = -r$, $j = -r$ and thus

$$\ell = j, g \equiv (xy)^j \pmod{\gamma_2(G)}, g \in \langle xy, \gamma_2(G) \rangle = M_3, \text{ that is, } \kappa(4) = 3.$$

The kernels of the third and fourth transfer are independent from the relational exponents $\alpha, \beta, \gamma, \delta, \rho$ and from the invariant k . \square

The following corollary 3.2.1 has been proved by Nebelung [23, p.200, Lem.6.7; p.204, Lem.6.10] without the use of explicit images of the transfers. We show that it can also be derived very easily from theorem 3.2 for groups of lower than second maximal class.

Corollary 3.2.1. *On the directed rooted tree of all metabelian 3-groups with abelianisation of type $(3, 3)$, every terminal group with invariant $k = 1$ has the same transfer type as its internal predecessor with invariant $k = 0$.*

Proof. Let G and H be two metabelian 3-groups with abelianisation of type $(3, 3)$, both of lower than second maximal class.

Assume that G has index of nilpotency $m = m(G) \geq 6$. If G has the invariant $k(G) = 1$, then G is represented by a terminal node in the tree [23] and its center $\zeta_1(G) = \gamma_{m-1}(G)$ is cyclic of order 3.

Suppose that $H \simeq G/\gamma_{m-1}(G)$ is the immediate predecessor of G and is therefore represented by an internal node in the tree. Then H has invariant $k(H) = 0$, index of nilpotency $m(H) = m - 1 \geq 5$, and the same invariant $e = e(H) = e(G) \geq 4$ as G .

If $(\alpha, \beta, \gamma, \delta, \rho)$ with $\rho = \pm 1$ are the relational exponents of G in equation (5), then those of H are given by $(\rho\delta, \beta, \rho\beta, \delta, 0)$, according to [23, p.184, Lem.5.2.6].

By theorem 3.2, we have the singulets of transfer types $\varkappa(3) = 4$ and $\varkappa(4) = 3$, for G as well as for H . Since the parameters (β, δ) of G and H are the same, we also have $\varkappa(2) = f(\beta, \delta)$, for both groups. Finally, the first singulet is given by $\varkappa(1) = f(\delta, \beta)$ for G with $k(G) = 1$, and by $\varkappa(1) = f(\rho\delta, \rho\beta)$ for H with $k(H) = 0$. However, since $\rho = \pm 1$, the values $f(\rho\delta, \rho\beta) = f(\delta, \beta)$ coincide.

This corollary is also valid for metabelian 3-groups of maximal and second maximal class [23]. \square

With the aid of Nebelung's isomorphism classes of metabelian 3-groups G [24] with representatives $G_\rho^{(m,n)}(\alpha, \beta, \gamma, \delta)$ of order 3^n and class $m - 1$, which satisfy the relations (5) with systems of exponents $(\alpha, \beta, \gamma, \delta, \rho)$, we now apply theorem 3.2 to the groups of lower than second maximal class with transfer types of section d, thereby distinguishing terminal and internal nodes on the tree of all isomorphism classes.

Theorem 3.3. *Let G be a metabelian 3-group with abelianisation of type $(3, 3)$ of lower than second maximal class, $e \geq 4$, and satisfying the assumptions of theorem 3.1 concerning the selection of the generators of $G = \langle x, y \rangle$ and the order of the maximal normal subgroups M_1, \dots, M_4 . Suppose that the transfer type of G is one of section d, which implies that G has the invariant $k = 0$.*

Then the position of G as a node on the directed rooted tree of all metabelian 3-groups with abelianisation of type $(3, 3)$ determines the exponents $\alpha, \beta, \gamma, \delta$ in the relations (5) and the canonical multiplet \varkappa of transfer types of G , which are given by table 4 for terminal nodes and by table 5 for internal nodes.

TABLE 4. Transfer types of section d for terminal groups

Type	$(\alpha, \beta, \gamma, \delta)$	\varkappa
d.19	(1, 0, 1, 0)	(4043)
d.19	(1, 0, -1, 0)	(3043)
d.23	(1, 0, 0, 0)	(1043)
d.25	(0, 0, 1, 0)	(2043)
d.25	(0, 0, -1, 0)	(2043)

TABLE 5. Transfer types of section d* for internal groups

Type	$(\alpha, \beta, \gamma, \delta)$	\varkappa
d*.19	(0, 1, 0, 1)	(0443)
d*.19	(0, -1, 0, 1)	(0343)
d*.23	(0, 0, 0, 1)	(0243)
d*.25	(0, 1, 0, 0)	(0143)
d*.25	(0, -1, 0, 0)	(0143)

Proof. The systems $(\alpha, \beta, \gamma, \delta)$ of relational exponents for groups of lower than second maximal class with transfer types of section d are given in the appendix of Nebelung's thesis [24, p.34, p.36, pp.60–68].

If we observe that $k = 0$, and thus $\rho = 0$, for the transfer types of section d, then the multiplets \varkappa of transfer types can be obtained immediately with the aid of theorem 3.2. \square

In the case of a quadratic base field, terminal and internal groups G with transfer types of section d can be distinguished by the parity of the index m of nilpotency of G .

Theorem 3.4. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic base field with elementary abelian bicyclic 3-class group of type (3, 3) and second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ having one of the transfer types of section d. Then the following statements hold.*

- (1) *The group G is represented by a terminal node on the tree, if and only if the index of nilpotency $m \geq 6$ and the invariant $e \geq 4$ are even.*
- (2) *The group G is represented by an internal node on the tree, if and only if the index of nilpotency $m \geq 7$ and the invariant $e \geq 5$ are odd.*
- (3) *K must be a real quadratic field and G must be of coclass $\text{cc}(G) \geq 3$.*

Proof. In this proof, we assume that the declarations of theorem 3.1 concerning the selection of the generators of $G = \langle x, y \rangle$ and the order of the maximal normal subgroups M_1, \dots, M_4 are satisfied. However, we point out that these assumptions do not appear in the statement of theorem 3.4.

We denote by N_1, \dots, N_4 the four unramified cyclic cubic extension fields of K , by L_1, \dots, L_4 their non-Galois absolutely cubic subfields, and we make use of proposition 5.4 in [19], which concerns the parity of the 3-exponents of the 3-class numbers of N_1, \dots, N_4 .

- (1) In the case of a terminal group G with transfer type d, we have $\kappa(2) = 0$, by theorem 3.3, and thus a total principalisation in N_2 . Therefore, $3^e = h_3(N_2) = h_3(L_2)^2$ with even exponent $e \geq 4$. Further, $1 \leq \kappa(1) \leq 4$ implies a partial principalisation in N_1 , and thus $3^{m-1} = h_3(N_1) = 3 \cdot h_3(L_1)^2$, since $k = 0$, with odd exponent $m - 1$ resp. even index of nilpotency $m \geq 6$, because $m = 4$ implies $e \leq m - 1 = 3$.
- (2) In the case of an internal group G with transfer type d^* , we have $1 \leq \kappa(2) \leq 4$, by theorem 3.3, and thus a partial principalisation in N_2 . Consequently, $3^e = h_3(N_2) = 3 \cdot h_3(L_2)^2$ with odd exponent $e \geq 5$, since for $e = 3$ only terminal groups with transfer type d are possible, by [24]. Further, $\kappa(1) = 0$ implies a total principalisation in N_1 , and thus $3^{m-1} = h_3(N_1) = h_3(L_1)^2$, since $k = 0$, with even exponent $m - 1$ resp. odd index of nilpotency $m \geq 7$, because $m = 5$ implies $e \leq m - 1 = 4$.
- (3) Since a total principalisation occurs in N_2 resp. N_1 for transfer types in both sections d and d^* , the base field K must be real quadratic, by proposition 5.3 in [19]. And since the invariant e has turned out to be at least equal to four, the group G must be of coclass $\text{cc}(G) \geq 3$.

□

Example 3.1. Occurrences of groups with transfer types in section d as second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of real quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ are extremely rare. Among the 2576 quadratic fields with 3-class group of type (3, 3) in the range $0 < D < 10^7$ of discriminants, there are only three cases with such groups [19, Ex.6.3, Tab.4]. We have

- a terminal group of transfer type d.23 with $e = 4$, $m = 6$, $n = 8$ for $D = 1\,535\,117$,
- a terminal group of transfer type d.19 with $e = 4$, $m = 6$, $n = 8$ for $D = 2\,328\,721$,
- an internal group of transfer type $d^*.25$ with $e = 5$, $m = 7$, $n = 10$ for $D = 8\,491\,713$.

3.3. Combinatorially possible transfer types of 3-groups. In this subsection, we arrange all combinatorially possible S_4 -orbits of the 5^4 quadruplets $\kappa \in [0, 4]^4$ by increasing cardinality of the image and decreasing number of fixed points. Table 6 shows the partial quadruplets and table 7 the total quadruplets as possible transfer types of metabelian 3-groups G with $G/\gamma_2(G)$ of type (3, 3), resp. principalisation types of base fields K with 3-class group $\text{Cl}_3(K)$ of type (3, 3). The orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\kappa) = (|\kappa^{-1}\{i\}|)_{0 \leq i \leq 4}$ the family of occupation numbers of the selected orbit representative κ and by $F = \{1 \leq i \leq 4 \mid \kappa(i) = i\}$ the set of fixed points of κ . In the characterising property, needed for equal numbers of fixed points, let $I = \{\kappa(i) \mid 1 \leq i \leq 4\}$ be the image of κ , $D = \kappa^{-1}(o(\kappa)^{-1}\{2\}) = \{i, j\}$ the preimage of a value occupied twice by κ , if $\kappa(i) = \kappa(j)$ for $1 \leq i < j \leq 4$, and $Z = \kappa^{-1}\{0\}$ the preimage of zero under κ .

If an orbit can be realised as a transfer type, then a suitable 3-group G is given, according to theorem 2.5 and [23, p.208, Satz 6.14]. In [18, p.80], the symbolic order has been given instead, that is the ideal of bivariate polynomials which annihilate the main commutator of G .

In table 6, the coarse classification into sections A to H is due to Scholz and Taussky [25]. The identification by ordinal numbers 1 to 19 and the set theoretical characterisation has been added in [18, p.80].

TABLE 6. The 19 S_4 -orbits of quadruplets $\varkappa \in [1, 4]^4$ with $\nu = 0$

Sec.	Nr.	repres. of orbit \varkappa	occupation numbers $o(\varkappa)$	fixed points $ F $	charact. property	cardinality of orbit $ \varkappa^{S_4} $	realising 3-group G
A	1	(1111)	(04000)	1	constant	4	$G_0^{(3)}(0, 1)$
B	2	(1211)	(03100)	2	almost	12	impossible
B	3	(1112)	(03100)	1	con-	24	impossible
H	4	(2111)	(03100)	0	stant	12	$G_1^{(5,6)}(1, 1, 1, 1)$
D	5	(1212)	(02200)	2		12	$G_0^{(4,5)}(1, 1, -1, 1)$
E	6	(1122)	(02200)	1		12	$G_0^{(m,m+1)}(1, -1, 1, 1)$
F	7	(2112)	(02200)	0		12	$G_0^{(m,m+e-2)}(1, 1, -1, 1)$
E	8	(1231)	(02110)	3		12	$G_0^{(m,m+1)}(1, 0, -1, 1)$
E	9	(1213)	(02110)	2		24	$G_0^{(m,m+1)}(0, 0, 1, 1)$
D	10	(1123)	(02110)	1	$D \setminus F \subset I$	24	$G_0^{(4,5)}(0, 0, -1, 1)$
F	11	(1321)	(02110)	1	$D \setminus F \not\subset I$	12	$G_0^{(m,m+e-2)}(1, 1, 0, 0)$
F	12	(3211)	(02110)	1	$D \cap F = \emptyset$	24	$G_0^{(m,m+e-2)}(1, 1, 0, -1)$
F	13	(2113)	(02110)	0	$D \subset I$	24	$G_0^{(m,m+e-2)}(1, 1, -1, 0)$
E	14	(2311)	(02110)	0	$D \not\subset I$	24	$G_0^{(m,m+1)}(0, -1, 1, 1)$
C	15	(1234)	(01111)	4	identity	1	impossible
G	16	(2134)	(01111)	2	transposition	6	$G_1^{(7,8)}(1, 0, 0, 1)$
C	17	(1342)	(01111)	1	3-cycle	8	impossible
C	18	(2341)	(01111)	0	4-cycle	6	impossible
G	19	(2143)	(01111)	0	2 disj. transp.	3	$G_1^{(5,6)}(0, -1, -1, 0)$
Total number:						256	

The 3-groups of sections A and D are determined uniquely. They are sporadic groups outside the periodic parts on the coclass graphs $\mathcal{G}(3, 1)$ resp. $\mathcal{G}(3, 2)$ [3, 17, 9, 8], stem groups of isoclinism families Φ_s in the sense of P. Hall [11], and coincide with the following groups in the listing of R. James [14, p.618 ff]: $G_0^{(3)}(0, 1) = \Phi_2(21)$ (extra special), $G_0^{(4,5)}(1, 1, -1, 1) = \Phi_6(221)_a$, $G_0^{(4,5)}(0, 0, -1, 1) = \Phi_6(221)_{c_2}$. The index of nilpotency for groups of section E is $m \geq 5$. For the groups of section F we have $m \geq 5$ and $e \geq 4$. The groups given for sections G and H occur as second 3-class groups of quadratic fields, but they are only special cases of families with two infinitely increasing parameters $m \geq 4$, $e \geq 3$, and $0 \leq k \leq 1$. The smallest members of these families are the stem groups $G_0^{(4,5)}(0, -1, -1, 0) = \Phi_6(221)_{d_0}$ and $G_0^{(4,5)}(1, 1, 1, 1) = \Phi_6(221)_{b_1}$, but they do not occur for quadratic base fields.

In table 7, the coarse classification into sections a to e is due to [23]. Additionally, we give an identification by ordinal numbers 1 to 26 and a set theoretical characterisation.

TABLE 7. The 26 S_4 -orbits of quadruplets $\varkappa \in [0, 4]^4 \setminus [1, 4]^4$ with $1 \leq \nu \leq 4$

Sec.	Nr.	repres. of orbit \varkappa	occupation numbers $o(\varkappa)$	fixed points $ F $	charact. property	cardinality of orbit $ \varkappa^{S_4} $	realising 3-group G
a	1	(0000)	(40000)	0	constant	1	$G_1^{(m)}(0, \pm 1)$, $m \geq 5$
a	2	(1000)	(31000)	1		4	$G_0^{(m)}(0, 1)$, $m \geq 4$
a	3	(0100)	(31000)	0		12	$G_0^{(m)}(\pm 1, 0)$, $m \geq 4$
e	4	(1100)	(22000)	1		12	impossible
e	5	(0110)	(22000)	0		12	impossible
e	6	(1200)	(21100)	2		6	impossible
e	7	(1020)	(21100)	1		24	impossible
e	8	(0012)	(21100)	0	$D \subset I$	12	impossible
e	9	(0120)	(21100)	0	$ D \cap I = 1$	24	impossible
b	10	(2100)	(21100)	0	$D \cap I = \emptyset$	6	$G_1^{(6,8)}(0, 0, 0, 0)$
e	11	(1110)	(13000)	1		12	impossible
e	12	(0111)	(13000)	0		4	impossible
e	13	(1210)	(12100)	2		24	impossible
e	14	(1120)	(12100)	1	$D \setminus F \subset I$	24	impossible
e	15	(1012)	(12100)	1	$D \setminus F \not\subset I$	24	impossible
e	16	(0211)	(12100)	1	$D \cap F = \emptyset$	12	impossible
e	17	(0112)	(12100)	0	$ D \cap I = 1$, $Z \subset I$	24	impossible
c	18	(2011)	(12100)	0	$D \cap I = \emptyset$, $Z \subset I$	12	$G_0^{(m,m+1)}(0, -1, 0, 1)$
d	19	(2110)	(12100)	0	$Z \not\subset I$	24	$G_0^{(m,m+e-2)}(1, 0, 1, 0)$
e	20	(1230)	(11110)	3	identity with 0	4	impossible
c	21	(1203)	(11110)	2		12	$G_0^{(m,m+1)}(0, 0, 0, 1)$
e	22	(1023)	(11110)	1	$Z \subset I$	24	impossible
d	23	(1320)	(11110)	1	$Z \not\subset I$	12	$G_0^{(m,m+e-2)}(1, 0, 0, 0)$
e	24	(0123)	(11110)	0	4-cycle with 0	24	impossible
d	25	(0321)	(11110)	0	2 disj. transp. with 0	12	$G_0^{(m,m+e-2)}(0, 1, 0, 0)$
e	26	(2310)	(11110)	0	$Z \not\subset I$	8	impossible
Total number:						369 = 625 - 256	

The groups with transfer type a.2 (resp. a.3) form families, starting with the sporadic stem groups $G_0^{(4)}(0, 1) = \Phi_3(211)_a$ (resp. $G_0^{(4)}(1, 0) = \Phi_3(211)_{b_1} = \text{Syl}_3 A_9$ and $G_0^{(4)}(-1, 0) = \Phi_3(211)_{b_2}$), and continuing with the stem groups $G_0^{(5)}(0, 1) = \Phi_9(2111)_a$ (resp. $G_0^{(5)}(1, 0) = \Phi_9(2111)_{b_0}$) and $G_0^{(6)}(0, 1) = \Phi_{35}(21111)_a$ (resp. $G_0^{(6)}(1, 0) = \Phi_{35}(21111)_{b_0}$ and $G_0^{(6)}(-1, 0) = \Phi_{35}(21111)_{b_1}$) on the periodic part of the coclass graph $\mathcal{G}(3, 1)$. Transfer type a.1 is additionally realised by the groups $C(3) \times C(3)$, $G_0^{(m)}(0, 0)$ with $m \geq 3$, and $G_1^{(m)}(0, 0)$ with $m \geq 5$. The smallest groups with $k = 1$ are $G_1^{(5)}(0, 0) = \Phi_{10}(1^5)$, $G_1^{(5)}(0, 1) = \Phi_{10}(2111)_{a_0}$, and $G_1^{(5)}(0, -1) = \Phi_{10}(2111)_{a_1}$. The index of nilpotency for groups of section c is $m \geq 4$. The smallest members of these two families are the stem groups $G_0^{(4,5)}(0, -1, 0, 1) = \Phi_6(221)_{d_1}$ and $G_0^{(4,5)}(0, 0, 0, 1) = \Phi_6(221)_{c_1}$. For groups of section d we have $m \geq 5$ and $e \geq 3$. The group given in section b occurs as second 3-class group of real quadratic fields, but it is only a special case of a family with two infinitely increasing parameters $m \geq 4$, $e \geq 3$, and $0 \leq k \leq 1$. The smallest member is $G_0^{(4,5)}(0, 0, 0, 0) = \Phi_6(1^5)$.

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